Compact locally maximal hyperbolic sets for smooth maps: fine statistical properties

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Abstract. Compact locally maximal hyperbolic sets are studied via geometrically defined functional spaces that take advantage of the smoothness of the map in a neighborhood of the hyperbolic set. This provides a self-contained theory that not only reproduces all the known classical results but gives also new insights on the statistical properties of these systems.

1. Introduction

The ergodic properties of uniformly hyperbolic maps can be described as follows. If T is a topologically mixing map on a compact locally maximal hyperbolic set Λ belonging to some smooth manifold X, and $\bar\phi : \Lambda \to \mathbb{R}$ is a Hölder continuous function, then there exists a unique probability measure $\mu_{\bar\phi}$ maximizing the variational principle with respect to $\bar\phi$ (the *Gibbs measure with potential* $\bar\phi$). Moreover, this measure enjoys strong statistical properties (exponential decay of correlations, central and local limit theorem...). When Λ is an attractor and the potential $\bar\phi$ is the jacobian of the map in the unstable direction (or, more generally, a function which is cohomologous to this one), then the measure $\mu_{\bar\phi}$ is the so-called SRB measure, which describes the asymptotic behavior of Lebesgue-almost every point in a neighborhood of Λ.

The proof of these results, due among others to Anosov, Margulis, Sinai, Ruelle, Bowen, is one of the main accomplishments of the theory of dynamical systems in the 70's. The main argument of their proof is to *code* the system, that is, to prove that it is semiconjugate to a subshift of finite type, to show the corresponding results for subshifts (first unilateral, and then bilateral), and to finally go back to the original system. These arguments culminate in Bowen's monograph [Bow75], where all the previous results are proved. Let us also mention another approach, using *specification*, which gives existence and uniqueness of Gibbs measures (but without exponential decay of correlations or limit theorems) through purely topological arguments [Bow74].

These methods and results have proved very fruitful for a manifold of problems. However, problems and questions of a new type have recently emerged, such as

- Strong statistical stability w.r.t. smooth or random perturbations;

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- Precise description of the correlations;
- Relationships between dynamical properties and the zeroes of the zeta function in a large disk.

It is possible to give partial answers to these questions using coding (see e.g. [Rue87, Hay90, Rue97, Pol03]), but their range is limited by the Hölder continuity of the foliation: the coding map can be at best Hölder continuous, and necessarily loses information on the smoothness properties of the transformation.

Recently, [BKL02] introduced a more geometric method to deal with these problems, in the case of the SRB measure. It was still limited by the smoothness of the foliation, but it paved the way to further progress. Indeed, Gouëzel and Liverani could get close to optimal answers to the first two questions (for the SRB measure of an Anosov map) in [GL06]. Baladi and Tsujii finally reached the optimal results in [Bal05, BT05] for these two questions (for the SRB measure of an hyperbolic attractor). A partial answer to the last question was first given in [Liv05] and a complete solution will appear in the paper [BT06]. See also the paper [LT06] for a very simple, although non optimal, argument.

The technical approach of these papers is as follows: they introduce spaces $\mathcal{B}$ of distributions, and an operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ with good spectral properties such that, for all smooth functions ψ_1, ψ_2 and all $n \in \mathbb{N}$,

$$\int \psi_1 \cdot \psi_2 \circ T^n \,\mathrm{dLeb} = \langle \mathcal{L}^n(\psi_1 \,\mathrm{dLeb}), \psi_2 \rangle. \tag{1.1}$$

The operator $\mathcal{L}$ has a unique fixed point, which corresponds to the SRB measure of the map T. The correlations are then given by the remaining spectral data of $\mathcal{L}$. In addition, abstract spectral theoretic arguments imply precise results on perturbations of T or zeta functions.

In this paper, we extend to the setting of Gibbs measures the results of [GL06]. This extension is not straightforward for the following reasons. First, the previous approaches for the SRB measure rely on the fact that there is already a reference measure to work with, the Lebesgue measure. For a general (yet to be constructed) Gibbs measure, there is no natural analogous of (1.1) which could be used to define the transfer operator $\mathcal{L}$. The technical consequence of this fact is that our space will not be a space of distribution on the whole space, rather a family of distributions on stable (or close to stable) leaves. Second, the SRB measure corresponds to a potential $\bar\phi_u$ – minus the logarithm of the unstable jacobian, with respect to some riemannian metric – which is in general *not* smooth, while we want our spaces to deal with very smooth objects. Notice however that $\bar\phi_u$ is cohomologous to a function which can be written as $\phi(x, E^s(x))$ where ϕ is a smooth function on the grassmannian of d_s dimensional subspaces of the tangent bundle $\mathcal{T}X$.[1] This is the kind of potential we will deal with.

The elements of our Banach space $\mathcal{B}$ will thus be objects "which can be integrated along small submanifolds of dimension d_s" (where d_s is the dimension of the stable manifolds). The first idea would be to take for $\mathcal{B}$ a space of differential forms of

[1] Take $\phi(x, E) = \log(\det DT(x)_{\restriction E}) - \log(\det DT(x))$, where det indicates the jacobian with respect to the given riemannian metric. Let $\bar\phi(x) = \phi(x, E^s(x))$ for $x \in \Lambda$. Since the angle between the stable and unstable direction is bounded from below, $\sum_{k=0}^{n-1}(\bar\phi \circ T^k - \bar\phi_u \circ T^k)$ is uniformly bounded on Λ. By Livsic theorem, this implies that $\bar\phi$ is cohomologous to $\bar\phi_u$. In particular, they give rise to the same Gibbs measure.

degree d_s. However, if α is such a form and ϕ is a potential as above, then $e^{\phi}\alpha$ is not a differential form any more. Hence, we will have to work with more general objects. Essentially, the elements of $\mathcal{B}$ are objects which associate, to any subspace E of dimension d_s of the tangent space, a volume form on E. Such an object can be integrated along d_s dimensional submanifolds, as required, and can be multiplied by e^{ϕ}. We define then an operator $\mathcal{L}$ on $\mathcal{B}$ by $\mathcal{L}\alpha = T_*(e^{\phi}\pi\alpha)$ where π is a truncation function (necessary to keep all the functions supported in a neighborhood of Λ, if Λ is not an attractor), and T_* denotes the (naturally defined) push-forward of an element of $\mathcal{B}$ under T. We will construct on $\mathcal{B}$ norms for which $\mathcal{L}$ has a good spectral behavior, in Section 2.

The main steps of our analysis are then the following.

(1) Prove a Lasota-Yorke inequality for $\mathcal{L}$ acting on $\mathcal{B}$, in Lemma 4.1 (by using the preliminary result in Lemma 3.4). This implies a good spectral description of $\mathcal{L}$ on $\mathcal{B}$: the spectral radius is some abstract quantity ϱ, yet to be identified, and the essential spectral radius is at most $\sigma\varrho$ for some small constant σ, related to the smoothness of the map. See Proposition 4.4 and Corollary 4.6.
(2) In this general setting, we analyze superficially the peripheral spectrum (that is, the eigenvalues of modulus ϱ), in Subsection 4.3. We prove that ϱ is an eigenvalue, and that there is a corresponding eigenfunction α_0 which induces a measure on d_s dimensional submanifolds (Lemma 4.9). This does not exclude the possibility of Jordan blocks or strange eigenfunctions.
(3) In the topologically mixing case, we check that α_0 is fully supported. By some kind of bootstrapping argument, this implies that $\|\mathcal{L}^n\| \leq C\varrho^n$, i.e., there is no Jordan block. Moreover, there is no other eigenvalue of modulus ϱ (Theorem 5.1).
(4) The adjoint of $\mathcal{L}$, acting on $\mathcal{B}'$, has an eigenfunction ℓ_0 for the eigenvalue ϱ. The linear form $\varphi \mapsto \ell_0(\varphi\alpha_0)$ is in fact a measure μ, this will be the desired Gibbs measure. Moreover, the correlations of μ are described by the spectral data of $\mathcal{L}$ acting on $\mathcal{B}$, as explained in Section 6.1.
(5) Finally, in Section 6.2 we prove that the dynamical balls have a very well controlled measure (bounded from below and above), see Proposition 6.3. This yields $\varrho = P_{\rm top}(\bar\phi)$ and the fact that μ is the unique equilibrium measure (Theorem 6.4).

It is an interesting issue to know whether there can indeed be Jordan blocks in the non topologically transitive case (this is not excluded by our results). The most interesting parts of the proof are probably the Lasota-Yorke estimate and the exclusion of Jordan blocks. Although the core of the argument is rather short and follows very closely the above scheme, the necessary presence of the truncation function induces several technical complications, which must be carefully taken care of and cloud a bit the overall logic. Therefore, the reader is advised to use the previous sketch of proof to find her way through the rigorous arguments. Note that the paper is almost completely self-contained, it only uses the existence and continuity of the stable and unstable foliation (and not their Hölder continuity nor their absolute continuity).

In addition, note that the present setting allows very precise answers to the first of the questions posed at the beginning of this introduction thanks to the possibility of applying the perturbation theory developed in [GL06, section 8] and based on [KL99]. Always in the spirit to help the reader we will give a flavor of such possibilities in Section 8 together with some obvious and less obvious examples to which our theory can be applied. In particular, in Proposition 8.1 we provide nice formulae for the derivative of the topological pressure and the Gibbs measure in the case of systems depending smoothly on a parameter.[2] Finally, a technical section (Section 9) on the properties of conformal leafwise measures is added both for completeness and because of its possible interest as a separate result.

Remark 1.1. *Let us point out that, although we follow the strategy of* [GL06], *similar results can be obtained also by generalizing the Banach spaces in* [BT05] *(M.Tsujii, private communication).*

To conclude the introduction let us give the description we obtain for the correlation functions. We consider an open set $U \subset X$ and a map $T \in \mathcal{C}^r(U, X)$,[3] diffeomorphic on its image (for some real $r > 1$). Suppose further that $\Lambda := \bigcap_{n\in\mathbb{Z}} T^n U$ is non empty and compact. Finally, assume that Λ is a hyperbolic set for T. Such a set is a *compact locally maximal hyperbolic set.* Let $\lambda > 1$ and $\nu < 1$ be two constants, respectively smaller than the minimal expansion of T in the unstable direction, and larger than the minimal contraction of T in the stable direction.

Denote by $\mathcal{W}^0$ the set of $\mathcal{C}^{r-1}$ function ϕ associating, to each $x \in U$ and each d_s dimensional subspace of the tangent space $\mathcal{T}_x X$ at x, an element of $\mathbb{R}$. Denote by $\mathcal{W}^1$ the set of $\mathcal{C}^r$ functions $\phi : U \to \mathbb{R}$. For $x \in \Lambda$, set $\bar\phi(x) = \phi(x, E^s(x))$ in the first case, and $\bar\phi(x) = \phi(x)$ in the second case. This is a Hölder continuous function on Λ. Assume that the restriction of T to Λ is topologically mixing.

Theorem 1.2. *Let $\phi \in \mathcal{W}^\iota$ for some $\iota \in \{0, 1\}$. Let $p \in \mathbb{N}^*$ and $q \in \mathbb{R}_+^*$ satisfy $p + q \le r - 1 + \iota$ and $q \ge \iota$. Let $\sigma > \max(\lambda^{-p}, \nu^q)$.[4] Then there exists a unique measure μ maximizing the variational principle for the potential $\bar\phi$,[5] and there exist a constant $C > 0$, a finite dimensional space F, a linear map $M : F \to F$ having a simple eigenvalue at 1 and no other eigenvalue with modulus ≥ 1, and two continuous mappings $\tau_1 : \mathcal{C}^p(U) \to F$ and $\tau_2 : \mathcal{C}^q(U) \to F'$ such that, for all $\psi_1 \in \mathcal{C}^p(U)$, $\psi_2 \in \mathcal{C}^q(U)$ and for all $n \in \mathbb{N}$,*

$$\left| \int \psi_1 \cdot \psi_2 \circ T^n \,\mathrm{d}\mu - \tau_2(\psi_2) M^n \tau_1(\psi_1) \right| \le C\sigma^n |\psi_2|_{\mathcal{C}^q(U)} |\psi_1|_{\mathcal{C}^p(U)}. \tag{1.2}$$

The coefficients of the maps τ_1 and τ_2 are therefore distributions of order at most p and q respectively, describing the decay of correlations of the functions. They extend the Gibbs distributions of [Rue87] to a higher smoothness setting.

When T is $\mathcal{C}^\infty$, we can take p and q arbitrarily large, and get a description of the correlations up to an arbitrarily small exponential error term. The SRB measure corresponds to a potential in $\mathcal{W}^0$, as explained above, and the restriction on p, q is

[2] Note that the formulae are in terms of exponentially converging sums, hence they can be easily used to actually compute the above quantities within a given precision.

[3] Here, and in the following, by $\mathcal{C}^r$ we mean the Banach space of functions continuously differentiable $\lfloor r\rfloor$ times, and with the $\lfloor r\rfloor$th derivative Hölder continuous of exponent $r - \lfloor r\rfloor$. Such a space is equipped with a norm $|\cdot|_{\mathcal{C}^r}$ such that $|fg|_{\mathcal{C}^r} \le |f|_{\mathcal{C}^r}|g|_{\mathcal{C}^r}$, that is $(\mathcal{C}^r, |\cdot|_{\mathcal{C}^r})$ is a Banach algebra. For example, if $r \in \mathbb{N}$, $|f|_{\mathcal{C}^r} := \sup_{k\le r} |f^{(k)}|_\infty 2^{r-k}$ will do.

[4] In fact, one can obtain better bounds by considering T^n, for large n, instead of T. We will not indulge on such subtleties to keep the exposition as simple as possible.

[5] Of course, this is nothing else than the classical Gibbs measure associated to the potential $\bar\phi$.

$p+q \leq r-1$, which corresponds to the classical Kitaev bound [Kit99].[6] Surprisingly, when the weight function belongs to $\mathcal{W}^1$, we can get up to $p+q=r$. In some sense, the results are better for maximal entropy measures than for SRB measures!

It is enlightening to consider our spaces for expanding maps, that is, when $d_s = 0$. In this case, "objects that can be integrated along stable manifolds" are simply objects assigning a value to a point, i.e., functions. Our Banach space $\mathcal{B}^{p,q}$ becomes the space of usual $\mathcal{C}^p$ functions, and we are led to the results of Ruelle in [Rue90].

Contents

2. The functional spaces

Consider a $\mathcal{C}^r$ differentiable manifold X. We start with few preliminaries.

2.1. **A touch of functional analysis.** To construct the functional spaces we are interested in, we will use an abstract construction that applies to each pair $\mathbb{V}, \Omega$, where $\mathbb{V}$ is a complex vector space and $\Omega \subset \mathbb{V}'$ is a subset of the (algebraic) dual with the property $\sup_{\ell \in \Omega} |\ell(h)| < \infty$ for each $h \in \mathbb{V}$. In such a setting we can define a seminorm on $\mathbb{V}$ by

$$\|h\| := \sup_{\ell \in \Omega} |\ell(h)|. \tag{2.1}$$

[6]In some cases, our bound is not optimal since p is restricted to be an integer.

If we call $\mathcal{B}$ the completion of $\mathbb{V}$ with respect to $\|\cdot\|$, we obtain a Banach space. Note that, by construction, Ω belongs to the unit ball of $\mathcal{B}'$. When $\|\cdot\|$ is a norm on $\mathbb{V}$, i.e., $\mathbb{V}_0 := \bigcap_{\ell\in\Omega} \ker \ell$ is reduced to $\{0\}$, then $\mathbb{V}$ can be identified as a subspace of $\mathcal{B}$. In general, however, there is only an inclusion of the algebraic quotient $\mathbb{V}/\mathbb{V}_0$ in $\mathcal{B}$.

2.2. **Differential geometry beyond forms.** Let $\mathcal{G}$ be the Grassmannian of d_s dimensional oriented subspaces of the tangent bundle $\mathcal{T}X$ to X. On it we can construct the complex line bundle $\mathcal{E} := \{(x, E, \omega) \,:\, (x, E) \in \mathcal{G}, \omega \in \bigwedge^{d_s} E' \otimes \mathbb{C}\}$. We can then consider the vector space $\mathcal{S}$ of the $\mathcal{C}^{r-1}$ sections of the line bundle $\mathcal{E}$. The point is that for each $\alpha \in \mathcal{S}$, each d_s dimensional oriented $\mathcal{C}^1$ manifold W and each $\varphi \in \mathcal{C}^0(W, \mathbb{C})$, we can define an integration of φ over W as if α was a usual differential form, by the formula

$$\ell_{W,\varphi}(\alpha) := \int_W \varphi\alpha := \int_U \varphi \circ \Phi(x) \Phi^* \alpha(\Phi(x), D\Phi(x)\mathbb{R}^{d_s}) \tag{2.2}$$

where $\Phi : U \to W$ is a chart and $\mathbb{R}^{d_s}$ is taken with the orientation determined by corresponding elements of the Grassmannian.[7] A direct computation shows that this definition is independent of the chart Φ, hence intrinsic.

Remark 2.1. *If ω is a d_s-differential form, then for each $(x, E) \in \mathcal{G}$ we can define $\alpha(x, E)$ to be the restriction of $\omega(x)$ to E. Thus the forms can be embedded in $\mathcal{S}$.*

Remark 2.2. *A Riemannian metric defines a volume form on any subspace of the tangent bundle of X. Thus, it defines an element of $\mathcal{S}$ with the property that its integral along any nonempty compact d_s dimensional submanifold is positive.*

2.2.1. *Integration of elements of $\mathcal{S}$.* If $f : \mathcal{G} \to \mathbb{C}$ is $\mathcal{C}^{r-1}$, then it is possible to multiply an element of $\mathcal{S}$ by f, to obtain a new element of $\mathcal{S}$. In particular, if $\alpha \in \mathcal{S}$, W is a d_s dimensional oriented $\mathcal{C}^1$ manifold and $\varphi \in \mathcal{C}^0(W, \mathbb{C})$, then there is a well defined integral

$$\int_W \varphi \cdot (f\alpha). \tag{2.3}$$

For $x \in W$, $\tilde{f}(x) := f(x, T_xW)$ is a continuous function on W and so is the function $\varphi\tilde{f}$. Hence, the integral

$$\int_W (\varphi\tilde{f}) \cdot \alpha \tag{2.4}$$

is also well defined. By construction, the integrals (2.3) and (2.4) coincide.

Convention 2.3. *We will write $\int_W \varphi f \alpha$ indifferently for these two integrals. More generally, implicitly, when we are working along a submanifold W, we will confuse f and $\tilde{f}$.*

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[7]If W cannot be covered by only one chart, then the definition is trivially extended, as usual, by using a partition of unity. Recall that, given a differential form ω on W and a base $\{e_i\}$ with its dual base $\{dx_i\}$ on $\mathbb{R}^{d_s}$, $\Phi^*\omega = \omega(D\Phi e_1, \dots, D\Phi e_{d_s})\, dx_1 \wedge \cdots \wedge dx_{d_s}$.

2.2.2. *Lie derivative of elements of $\mathcal{S}$.* If ϕ is a local diffeomorphism of X, it can be lifted through its differential to a local bundle isomorphism of $\mathcal{E}$. Hence, if $\alpha \in \mathcal{S}$, its pullback $\phi^*\alpha$ is well defined. In a pedestrian way, an element of $\mathcal{S}$ is a function from $\mathcal{F} := \{(x, E, e_1 \wedge \cdots \wedge e_{d_s}) \,:\, (x, E) \in \mathcal{G}, e_1, \dots, e_{d_s} \in E\}$ to $\mathbb{C}$, satisfying the homogeneity relation $\alpha(x, E, \lambda e_1 \wedge \cdots \wedge e_{d_s}) = \lambda \alpha(x, E, e_1 \wedge \cdots \wedge e_{d_s})$. If $(x, E) \in \mathcal{G}$ and $e_1, \dots, e_{d_s}$ is a family of vectors in E, then $\phi^*\alpha$ is given by

$$(\phi^*\alpha)(x, E, e_1 \wedge \cdots \wedge e_{d_s}) = \alpha(\phi(x), D\phi(x)E, D\phi(x)e_1 \wedge \cdots \wedge D\phi(x)e_{d_s}). \tag{2.5}$$

Given a vector field v, we will write L_v for its Lie derivative. Given a $\mathcal{C}^k$ vector field v on X, with $k \geq 1$, there is a canonical way to lift it to a $\mathcal{C}^{k-1}$ vector field on $\mathcal{F}$, as follows. Let ϕ_t be the flow of the vector field v. For $\alpha \in \mathcal{S}$, the pullback $\phi_t^*\alpha$ is well defined. The quantity $\left.\frac{\mathrm{d}\phi_t^*\alpha}{\mathrm{d}t}\right|_{t=0}$ is then given by the Lie derivative of α against a $\mathcal{C}^{k-1}$ vector field, which we denote by $v^{\mathcal{F}}$. The following result will be helpful in the following:

Proposition 2.4 ([KMS93], Lemma 6.19)**.** *The map $v \mapsto v^{\mathcal{F}}$ is linear. Moreover, if v_1, v_2 are two $\mathcal{C}^2$ vector fields on X,*

$$[L_{v_1^{\mathcal{F}}}, L_{v_2^{\mathcal{F}}}] = L_{[v_1, v_2]^{\mathcal{F}}}. \tag{2.6}$$

Remark 2.5. *We will use systematically the above proposition to confuse v and $v^{\mathcal{F}}$, so in the following we will suppress the superscript $\mathcal{F}$, where this does not create confusion.*

If W is a compact submanifold of X with boundary, and $q \in \mathbb{R}_+$, we will write $\mathcal{C}_0^q(W)$ for the set of $\mathcal{C}^q$ functions from W to $\mathbb{C}$ vanishing on the boundary of W, and $\mathcal{V}^q(W)$ for the set of $\mathcal{C}^q$ vector fields defined on a neighborhood of W in X.

If $v \in \mathcal{V}^1(W)$ is tangent to W along W, and $\alpha \in \mathcal{S}$, then $L_v\alpha$ can also be obtained along W by considering the restriction of α to W, which is a volume form, and then taking its (usual) Lie derivative with respect to the restriction of v to W. Therefore, the usual Stokes formula still applies in this context, and gives the following integration by parts formula.

Proposition 2.6. *Let W be a compact submanifold with boundary of dimension d_s, let $\alpha \in \mathcal{S}$, let $v \in \mathcal{V}^1(W)$ be tangent to W along W, and let $\varphi \in \mathcal{C}_0^1(W)$. Then*

$$\int_W \varphi L_v \alpha = -\int_W (L_v\varphi)\alpha. \tag{2.7}$$

2.3. **The norms.** Let Σ be a set of d_s dimensional compact $\mathcal{C}^r$ submanifolds of X, with boundary. To such a Σ, we will associate a family of norms on $\mathcal{S}$ as follows.

Definition 2.7. *A triple $(t, q, \iota) \in \mathbb{N} \times \mathbb{R}_+ \times \{0, 1\}$ is* correct *if $t + q \leq r - 1 + \iota$, and $q \geq \iota$ or $t = 0$.*

Remark 2.8. *Notice that, if (t, q, ι) is correct and $t \geq 1$, then $(t-1, q+1, \iota)$ is also correct.*

For any correct (t, q, ι), consider the set

$$\begin{aligned}\Omega_{t,q+t,\iota} = \{(W, \varphi, v_1, \dots, v_t) \,:\, & W \in \Sigma, \varphi \in \mathcal{C}_0^{q+t}(W) \text{ with } |\varphi|_{\mathcal{C}^{q+t}(W)} \leq 1, \\ & v_1, \dots, v_t \in \mathcal{V}^{q+t-\iota}(W) \text{ with } |v_i|_{\mathcal{V}^{q+t-\iota}(W)} \leq 1\}.\end{aligned}$$

To each $\omega \in \Omega_{t,q+t,\iota}$, we can associate a linear form ℓ_ω on $\mathcal{S}$, by

$$\ell_\omega(\alpha) = \int_W \varphi \cdot L_{v_1} \dots L_{v_t}(\alpha). \tag{2.8}$$

Indeed, this is clearly defined if $t = 0$. Moreover, if $t > 0$, the vector field v_t is $\mathcal{C}^{q+t-\iota}$, and is in particular $\mathcal{C}^1$. Hence, the lifted vector field $v_t^{\mathcal{F}}$ is well defined, and $L_{v_t}(\alpha) \in \mathcal{C}^{\min(r-2,q+t-\iota-1)} = \mathcal{C}^{q+t-\iota-1}$ since $q+t-\iota \leq r-1$. Going down by induction, we have in the end $L_{v_1} \dots L_{v_t}(\alpha) \in \mathcal{C}^{q-\iota}$, which does not create any smoothness problem since $q \geq \iota$.

We can then define the seminorms

$$\|\alpha\|^-_{t,q+t,\iota} := \sup_{\omega \in \Omega_{t,q+t,\iota}} \ell_\omega(\alpha). \tag{2.9}$$

For $p \in \mathbb{N}$, $q \geq 0$ and $\iota \in \{0,1\}$ such that (p,q,ι) is correct, we define then

$$\|\alpha\|_{p,q,\iota} := \sum_{t=0}^{p} \|\alpha\|^-_{t,q+t,\iota}. \tag{2.10}$$

We will use the notation $\mathcal{B}^{p,q,\iota}$ for the closure of $\mathcal{S}$ in the above seminorm. This construction is as described in Section 2.1.

Note that (2.10) defines in general only a seminorm on $\mathcal{S}$. Indeed, if $\alpha \in \mathcal{S}$ vanishes in a neighborhood of the tangent spaces to elements of Σ, then $\|\alpha\|_{p,q,\iota} = 0$.

3. The dynamics

In Section 2 the dynamics did not play any role, yet all the construction depends on the choice of Σ. In fact, such a choice encodes in the geometry of the space the relevant properties of the dynamics. In this chapter we will first define Σ by stating the relevant properties it must enjoy, then define the transfer operator and study its properties when acting on the resulting spaces.

3.1. Admissible leaves.

Recall from the introduction that we have an open set $U \subset X$ and a map $T \in \mathcal{C}^r(U, X)$, diffeomorphic on its image. Furthermore $\Lambda := \bigcap_{n\in\mathbb{Z}} T^n U$ is non empty and compact and Λ is a hyperbolic set for T. In addition, once and for all, we fix an open neighborhood U' of Λ, with compact closure in U, such that $TU' \subset U$ and $T^{-1}U' \subset U$, and small enough so that the restriction of T to U' is still hyperbolic. For $x \in U'$, denote by $C_s(x)$ the stable cone at x. Let finally V be a small neighborhood of Λ, compactly contained in U'.

Definition 3.1. *A set Σ of d_s dimensional compact submanifolds of U' with boundary is an* admissible set of leaves *if*

1. *Each element W of Σ is a $\mathcal{C}^r$ submanifold of X, its tangent space at $x \in W$ is contained in $C_s(x)$, and $\sup_{W\in\Sigma} |W|_{\mathcal{C}^r} < \infty$. Moreover, for any point x of Λ, there exists $W \in \Sigma$ containing x and contained in $W^s(x)$. Additionally, $\sup_{W\in\Sigma} \operatorname{diam}(W) < \infty$, and there exists $\varepsilon > 0$ such that each element of Σ contains a ball of radius ε. Moreover, to each leaf $W \in \Sigma$ intersecting V, we associate an* enlargement *W^e of W, which is the union of a uniformly bounded number of leaves $W_1, \dots, W_k \in \Sigma$, containing W, and such that $\operatorname{dist}(\partial W, \partial W^e) > 2\delta_0$ for some $\delta_0 > 0$ (independent of W).*
2. *Let us say that two leaves $W, W' \in \Sigma$ are (C,ε)-close if there exists a $\mathcal{C}^{r-1}$ vector field v, defined on a neighborhood of W, with $|v|_{\mathcal{C}^{r-1}} \leq \varepsilon$, and such that its flow ϕ_t is uniformly bounded in $\mathcal{C}^r$ by C and satisfies*

$\phi_1(W) = W'$ and $\phi_t(W) \in \Sigma$ for $0 \le t \le 1$. We assume that there exists a constant C_Σ such that, for all $\varepsilon > 0$, there exists a finite number of leaves $W_1, \dots, W_k \in \Sigma$ such that any $W \in \Sigma$ is (C_Σ, ε)-close to a leaf W_i with $1 \le i \le k$.

(3) *There exist $C > 0$ and a sequence ε_n going exponentially fast to 0 such that, for all $n \in \mathbb{N}^*$, for all $W \in \Sigma$, there exist a finite number of leaves $W_1, \dots, W_k \in \Sigma$ and $\mathcal{C}^r$ functions $\rho_1, \dots, \rho_k$ with values in $[0,1]$ compactly supported on W_i, with $|\rho_i|_{\mathcal{C}^r(W_i)} \le C$, and such that the set $W^{(n)} = \{x \in W, \forall\, 0 \le i \le n-1, T^{-i}x \in V\}$ satisfies: $T^{-n}W^{(n)} \subset \bigcup W_i$, and $\sum \rho_i = 1$ on $T^{-n}W^{(n)}$, and any point of $T^{-n}W^{(n)}$ is contained in at most C sets W_i. Moreover, W_i is $(C_\Sigma, \varepsilon_n)$-close to an element of Σ contained in the stable manifold of a point of Λ. Finally, $T^n\left(\bigcup_{i=1}^k W_i\right)$ is contained in the enlargement W^e of W, and even in the set $\{x \in W^e \ :\ \mathrm{dist}(x, \partial W^e) > \delta_0\}$.*

The first property of the definition means that the elements of Σ are close to stable leaves in the $\mathcal{C}^1$ topology, and have a reasonable size. The second condition means that there are sufficiently many leaves, and will imply some compactness properties. The third property is an invariance property and means that we can iterate the leaves backward.

In [GL06], the existence of admissible sets of leaves is proved for Anosov systems. The proof generalizes in a straightforward way to this setting. Hence, the following proposition holds.

Proposition 3.2. *Admissible sets of leaves do exist.*

We choose once and for all such an admissible set of leaves, and denote it by Σ.

3.2. **Definition of the Operator.** We will consider the action of the composition by T on the previously defined spaces. For historical reasons, we will rather consider the composition by T^{-1}, but this choice is arbitrary. To keep the functions supported in U, we need a truncation function. Let π be a $\mathcal{C}^r$ function taking values in $[0,1]$, equal to 1 on a neighborhood of Λ and compactly supported in $T(V)$.

We need also to introduce a *weight*. We will consider two classes of weights. Let $\mathcal{W}^0$ be the set of $\mathcal{C}^{r-1}$ functions ϕ from $\mathcal{G}$ to $\mathbb{R}$, such that if $x \in U$ and F and F' are the same subspace of T_xU but with opposite orientations, then $\phi(F) = \phi(F')$. This condition makes it possible to define a function $\bar\phi$ on Λ by $\bar\phi(x) = \phi(x, E^s(x))$ (where the orientation of $E^s(x)$ is not relevant by the previous property). Let $\mathcal{W}^1$ be the set of $\mathcal{C}^r$ functions from X to $\mathbb{R}$. Of course, an element of $\mathcal{W}^1$ is an element of $\mathcal{W}^0$ as well. Yet, slightly stronger results hold true for weights in $\mathcal{W}^1$.

For each truncation function π, and each weight $\phi \in \mathcal{W}^\iota$, $\iota \in \{0,1\}$, we define a *truncated and weighted transfer operator* (or simply transfer operator) $\mathcal{L}_{\pi,\phi} : \mathcal{S} \to \mathcal{S}$ by

$$\mathcal{L}_{\pi,\phi}\alpha(x,E) := \pi(T^{-1}x) e^{\phi(T^{-1}x, DT^{-1}(x)E)} T_*\alpha(T^{-1}x, DT^{-1}(x)E). \tag{3.1}$$

In terms of the action of diffeomorphisms on elements of $\mathcal{S}$ defined in (2.5), this formula can be written as $\mathcal{L}_{\pi,\phi}\alpha = T_*(\pi e^\phi \alpha)$. It is clear that an understanding of the iterates of $\mathcal{L}_{\pi,\phi}$ would shed light on the mixing properties of T. The operator $\mathcal{L}_{\pi,\phi}$ does not have good asymptotic properties on $\mathcal{S}$ with its $\mathcal{C}^{r-1}$ norm, but we will show that it behaves well on the spaces $\mathcal{B}^{p,q,\iota}$.

If W is a submanifold of dimension d_s contained in U', φ is a continuous function on W with compact support and $\alpha \in \mathcal{S}$, then by definition

$$\int_W \varphi \mathcal{L}_{\pi,\phi}\alpha = \int_{T^{-1}W} \varphi \circ T \pi e^{\phi} \alpha. \tag{3.2}$$

Recall that this integral is well defined by Convention 2.3.

3.3. **Main dynamical inequality.** When (p, q, ι) is correct, i.e., $p+q-\iota \leq r-1$, and $q \geq \iota$ or $t = 0$, and the weight ϕ belongs to $\mathcal{W}^\iota$, we can study the spectral properties of $\mathcal{L}_{\pi,\phi}$ acting on $\mathcal{B}^{p,q,\iota}$. Notice that, for a weight belonging to $\mathcal{W}^1$, this means that we can go up to $p+q = r$, i.e., we can reach the differentiability of the map. Before proceeding we need a definition.

Definition 3.3. *For each $W \in \Sigma$ and $n \in \mathbb{N}$ let $\{W_j\}$ be any covering of $T^{-n}W^{(n)}$ as given by the third item in Definition 3.1. We define*

$$\varrho_n := \left(\sup_{W\in\Sigma} \sum_j \left| e^{S_n\phi}\pi_n \right|_{\mathcal{C}^{r-1+\iota}(W_j)} \right)^{1/n}, \tag{3.3}$$

where $\pi_n := \prod_{k=0}^{n-1} \pi \circ T^k$ and, for each function $f : X \to \mathbb{C}$, $S_n f := \sum_{k=0}^{n-1} f \circ T^k$. Here, to define $S_n\phi$ along W, we use Convention 2.3.[8]

The main lemma to prove Lasota-Yorke type inequalities is the following:

Lemma 3.4. *Let $t \in \mathbb{N}$ and $q \geq 0$. If (t, q, ι) is correct, there exist constants $C > 0$ and $C_n > 0$ for $n \in \mathbb{N}$ such that, for any $\alpha \in \mathcal{S}$,*

$$\left\| \mathcal{L}^n_{\pi,\phi}\alpha \right\|^-_{t,q+t,\iota} \leq C\varrho_n^n \lambda^{-tn} \left\|\alpha\right\|^-_{t,q+t,\iota} + C_n \sum_{0\leq t'<t} \left\|\alpha\right\|^-_{t',q+t',\iota}. \tag{3.4}$$

Moreover, if $(t, q+1, \iota)$ is also correct,

$$\begin{aligned} \left\| \mathcal{L}^n_{\pi,\phi}\alpha \right\|^-_{t,q+t,\iota} \leq C\varrho_n^n \nu^{(q+t)n} \lambda^{-tn} \left\|\alpha\right\|^-_{t,q+t,\iota} \\ + C_n \sum_{0\leq t'<t} \left\|\alpha\right\|^-_{t',q+t',\iota} + C_n \left\|\alpha\right\|^-_{t,q+t+1,\iota}. \end{aligned} \tag{3.5}$$

Proof. Take $\omega = (W, \varphi, v_1, \dots, v_t) \in \Omega_{t,q+t,\iota}$. Let ρ_j be an adapted partition of unity on $T^{-n}W^{(n)}$, as given in (3) of Definition 3.1. We want to estimate

$$\int_W \varphi \cdot L_{v_1} \dots L_{v_t}(\mathcal{L}^n_{\pi,\phi}\alpha) = \sum_j \int_{W_j} \varphi \circ T^n \rho_j \cdot L_{w_1} \dots L_{w_t}(\alpha \cdot e^{S_n\phi} \prod_{k=0}^{n-1} \pi \circ T^k), \tag{3.6}$$

where $w_i = (T^n)^*(v_i)$. Remembering that $\pi_n := \prod_{k=0}^{n-1} \pi \circ T^k$,

$$L_{w_1} \dots L_{w_t}(\alpha \cdot e^{S_n\phi}\pi_n) = \sum_{A\subset\{1,\dots,t\}} \left(\prod_{i\notin A} L_{w_i} \right)(\alpha) \cdot \left(\prod_{i\in A} L_{w_i} \right)(e^{S_n\phi}\pi_n). \tag{3.7}$$

[8]Note that the volume of $T^{-n}W$ grows at most exponentially. Thus, given the condition (3) of Definition 3.1 on the bounded overlap of the W_j, the cardinality of $\{W_j\}$ can grow at most exponentially as well. In turn, this means that there exists a constant C such that $\varrho_n \leq C$.

We claim that, for any $A \subset \{1, \dots, t\}$,

$$\left(\prod_{i\in A} L_{w_i}\right)(e^{S_n\phi}\pi_n) \in \mathcal{C}^{q+t-\#A}. \tag{3.8}$$

Assume first that $\iota = 0$. Then $q + t \le r - 1$. The lift $w_k^{\mathcal{F}}$ of any of the vector fields w_k is in $\mathcal{C}^{q+t-1}$, hence $L_{w_k}(e^{S_n\phi}\pi_n) \in \mathcal{C}^{\min(r-2,q+t-1)} = \mathcal{C}^{q+t-1}$. Equation (3.8) then follows inductively on $\#A$. On the other hand, if $\iota = 1$, the vector field w_k is only $\mathcal{C}^{q+t-1}$ (and so $w_k^{\mathcal{F}}$ is only $\mathcal{C}^{q+t-2}$, which is not sufficient). However, there is no need to lift the vector field w_k to $\mathcal{F}$ since ϕ is defined on X. Hence, we get $L_{w_k}(e^{S_n\phi}\pi_n) \in \mathcal{C}^{\min(r-1,q+t-1)} = \mathcal{C}^{q+t-1}$. Equation (3.8) easily follows.

In the right hand side of (3.6), we can use (3.7) to compute $L_{w_1} \dots L_{w_t}(\alpha \cdot e^{S_n\phi}\pi_n)$. Any term with $A \neq \emptyset$ is then estimated by $C_n \|\alpha\|^-_{t-\#A,q+t-\#A,\iota}$, thanks to (3.8). Hence, to conclude, it suffices to estimate the remaining term with $A = \emptyset$:

$$\int_{W_j} \varphi \circ T^n \rho_j e^{S_n\phi}\pi_n \cdot L_{w_1} \dots L_{w_t}(\alpha). \tag{3.9}$$

To this end we decompose w_i as $w_i^u + w_i^s$ where w_i^u and w_i^s are $\mathcal{C}^{q+t-\iota}$ vector fields, w_i^s is tangent to W_j, and $|w_i^u|_{\mathcal{C}^{q+t-\iota}} \le C\lambda^{-n}$.[9] Clearly $L_{w_i} = L_{w_i^u} + L_{w_i^s}$. Hence, for $\sigma \in \{s,u\}^t$, we must study the integrals

$$\int_{W_j} \varphi \circ T^n \rho_j e^{S_n\phi}\pi_n \cdot L_{w_1^{\sigma_1}} \dots L_{w_t^{\sigma_t}}(\alpha). \tag{3.10}$$

Notice that, if we exchange two of these vector fields, the difference is of the form $\int_{W_j} \tilde{\varphi} L_{w'_1} \dots L_{w'_{t-1}}(\alpha)$ where $w'_1, \dots, w'_{t-1}$ are $\mathcal{C}^{q+t-1-\iota}$ vector fields. Indeed, $L_w L_{w'} = L_{w'} L_w + L_{[w,w']}$ by Proposition 2.4, and $[w,w']$ is a $\mathcal{C}^{q+t-1-\iota}$ vector field. In particular, up to $C_n \|\alpha\|^-_{t-1,q+t-1,\iota}$, we can freely exchange the vector fields.

Suppose first that $\sigma_1 = s$. Then, by (2.7), the integral (3.10) is equal to

$$-\int_{W_j} L_{w_1^s}(\varphi \circ T^n \rho_j e^{S_n\phi}\pi_n) \cdot L_{w_2^{\sigma_2}} \dots L_{w_t^{\sigma_t}}(\alpha). \tag{3.11}$$

This is bounded by $C_n \|\alpha\|^-_{t-1,q+t-1,\iota}$. More generally, if one of the σ_i's is equal to s, we can first exchange the vector fields as described above to put the corresponding $L_{w_i^s}$ in the first place, and then integrate by parts. Finally, we have

$$\begin{aligned} \int_{W_j} \varphi \circ T^n \rho_j e^{S_n\phi}\pi_n \cdot L_{w_1} \dots L_{w_t}(\alpha) \\ = \int_{W_j} \varphi \circ T^n \rho_j e^{S_n\phi}\pi_n \cdot L_{w_1^u} \dots L_{w_t^u}(\alpha) + \mathcal{O}(\|\alpha\|^-_{t-1,q+t-1,\iota}). \end{aligned} \tag{3.12}$$

[9] Such a decomposition is achieved in [GL06, Appendix A] (and the computation is even easier since the smoothing is not required). The argument roughly goes as follows. Consider a $\mathcal{C}^r$ foliation transverse to W_j, and push it by T^n. Around $T^n W_j$, consider also a foliation given by translates (in some chart with uniformly bounded $\mathcal{C}^r$ norm) of $T^n W_j$. Then project simply v_i on these two transverse foliations, and pull everything back under T^n. This is essentially the desired decomposition.

We are now positioned to prove (3.4). The last integral in (3.12) is bounded by

$$\left|\varphi\circ T^n\rho_j e^{S_n\phi}\pi_n\right|_{\mathcal{C}^{q+t}(W_j)}\prod_{i=1}^{t}|w_i^u|_{\mathcal{C}^{q+t-\iota}(W_j)}\,\|\alpha\|^-_{t,q+t,\iota} \\ \leq C\lambda^{-tn}\left|e^{S_n\phi}\pi_n\right|_{\mathcal{C}^{q+t}(W_j)}\|\alpha\|^-_{t,q+t,\iota}\,. \tag{3.13}$$

Summing the inequalities (3.13) over j and remembering Definition 3.3 yields (3.4).

This simple argument is not sufficient to prove (3.5), since we want also to gain a factor $\nu^{-(q+t)n}$ (if we are ready to pay the price of having a term $\|\alpha\|^-_{t,q+t+1,\iota}$ in the upper bound). To do this, we will smoothen the test function φ. Let $\mathbb{A}_\varepsilon\varphi$ be obtained by convolving φ with a mollifier of size ε. If a is the largest integer less than $q+t$, we have $|\mathbb{A}_\varepsilon\varphi-\varphi|_{\mathcal{C}^a}\leq C\varepsilon^{q+t-a}$, the function $\mathbb{A}_\varepsilon\varphi$ is bounded in $\mathcal{C}^{q+t}$ independently of ε, and it belongs to $\mathcal{C}^{q+t+1}$. We choose $\varepsilon=\nu^{(q+t)n/(q+t-a)}$. In this way,

$$|(\varphi-\mathbb{A}_\varepsilon\varphi)\circ T^n|_{\mathcal{C}^{q+t}(W_j)}\leq C\nu^{(q+t)n}. \tag{3.14}$$

Then (3.12) implies

$$\int_{W_j}\varphi\circ T^n\rho_j e^{S_n\phi}\pi_n\cdot L_{w_1^u}\dots L_{w_t^u}(\alpha) \\ =\int_{W_j}(\varphi-\mathbb{A}_\varepsilon\varphi)\circ T^n\rho_j e^{S_n\phi}\pi_n\cdot L_{w_1^u}\dots L_{w_t^u}(\alpha) \\ +\int_{W_j}(\mathbb{A}_\varepsilon\varphi)\circ T^n\rho_j e^{S_n\phi}\pi_n\cdot L_{w_1^u}\dots L_{w_t^u}(\alpha).$$

The last integral is bounded by $C_n\|\alpha\|^-_{t,q+t+1,\iota}$. And the previous one is at most

$$\left|(\varphi-\mathbb{A}_\varepsilon\varphi)\circ T^n\rho_j e^{S_n\phi}\pi_n\right|_{\mathcal{C}^{q+t}(W_j)}\prod_{i=1}^{t}|w_i^u|_{\mathcal{C}^{q+t-\iota}(W_j)}\,\|\alpha\|^-_{t,q+t,\iota} \\ \leq C\nu^{(q+t)n}\left|e^{S_n\phi}\pi_n\right|_{\mathcal{C}^{q+t}(W_j)}\lambda^{-tn}\|\alpha\|^-_{t,q+t,\iota}\,.$$

Summing over j and remembering Definition 3.3, we finally have (3.5). □

4. Spectral properties of the Transfer Operator

In this section we investigate the spectral radius and the essential spectral radius of the Ruelle operator. We will use constants $\bar\varrho>0$ and $d\in\mathbb{N}$ such that[10]

$$\exists C>0,\forall n\in\mathbb{N}^*,\quad \varrho_n^n\leq Cn^d\bar\varrho^n. \tag{4.1}$$

4.1. **Quasi compactness.** As usual, the proof of the quasi compactness of the transfer operator is based on two ingredients: a Lasota-Yorke type inequality and a compact embedding between spaces. See [Bal00] if unfamiliar with such ideas.

[10]Such constants do exist, see footnote 8.

4.1.1. *Lasota-Yorke inequality.*

Lemma 4.1. *Let $\iota \in \{0,1\}$. For all $p \in \mathbb{N}$ and $q \geq 0$ such that (p,q,ι) is correct, for all $(\bar{\varrho}, d)$ satisfying (4.1), there exists a constant $C > 0$ such that, for all $n \in \mathbb{N}^*$, for all $\alpha \in \mathcal{S}$,*

$$\left\|\mathcal{L}_{\pi,\phi}^n \alpha\right\|_{p,q,\iota} \leq C n^d \bar{\varrho}^n \left\|\alpha\right\|_{p,q,\iota}. \tag{4.2}$$

Moreover, if $p > 0$, the following inequality also holds:

$$\left\|\mathcal{L}_{\pi,\phi}^n \alpha\right\|_{p,q,\iota} \leq C \bar{\varrho}^n \max(\lambda^{-p}, \nu^q)^n \left\|\alpha\right\|_{p,q,\iota} + C n^d \bar{\varrho}^n \left\|\alpha\right\|_{p-1,q+1,\iota}. \tag{4.3}$$

Finally, if $p \geq 0$, there exists $\sigma < 1$ (independent of $\bar{\varrho}$ and d) such that

$$\left\|\mathcal{L}_{\pi,\phi}^n \alpha\right\|_{p,q,\iota} \leq C \bar{\varrho}^n \sigma^n \left\|\alpha\right\|_{p,q,\iota} + C n^d \bar{\varrho}^n \left\|\alpha\right\|_{0,q+p,\iota}. \tag{4.4}$$

Proof. The inequality (4.4) is an easy consequence of (4.3) and an induction on p. Moreover, (4.2) for $p = 0$ is a direct consequence of Equation (3.4) with $t = 0$, and (4.1). Note also that (4.3) for $p > 0$ implies (4.2) for the same p. Hence, it is sufficient to prove that (4.2) at $p-1$ implies (4.3) at p.

Choose any $\lambda' > \lambda$ and $\nu' < \nu$ respectively smaller and larger than the best expansion and contraction constants of T in the unstable and stable direction. Lemma 3.4 still applies with λ' and μ' instead of λ and μ. Hence, there exist constants C_0 and C'_n such that, for all $0 \leq t \leq p$, and setting $\sigma_1 := \max(\lambda'^{-p}, \nu'^q)$,

$$\left\|\mathcal{L}_{\pi,\phi}^n \alpha\right\|_{t,q+t,\iota}^- \leq C_0 n^d \bar{\varrho}^n \sigma_1^n \left\|\alpha\right\|_{t,q+t,\iota}^- + C'_n \sum_{t'<t} \left\|\alpha\right\|_{t',q+t',\iota}^- + C'_n \left\|\alpha\right\|_{p-1,q+1,\iota}.$$

To prove this, we use (3.4) for $t = p$, and (3.5) for $t < p$ (in which case $\left\|\alpha\right\|_{t,q+t+1,\iota}^- \leq \left\|\alpha\right\|_{p-1,q+1,\iota}$).

Let $\sigma = \max(\lambda^{-p}, \nu^q)$. There exists N such that $C_0 N^d \sigma_1^N \leq \sigma^N/2$. We fix it once and for all. Fix also once and for all a large constant $K > 2$ such that $\frac{C'_N K^{-1}}{1-K^{-1}} \leq \bar{\varrho}^N \sigma^N/2$, and define a new seminorm on $\mathcal{S}$ by $\left\|\alpha\right\|'_{p,q,\iota} = \sum_{t=0}^p K^{-t} \left\|\alpha\right\|_{t,q+t,\iota}^-$. Then $\left\|\mathcal{L}_{\pi,\phi}^N \alpha\right\|'_{p,q,\iota}$ is at most

$$\begin{aligned}
&\sum_{t=0}^p K^{-t} \left(\bar{\varrho}^N (\sigma^N/2) \left\|\alpha\right\|_{t,q+t,\iota}^- + C'_N \sum_{t'<t} \left\|\alpha\right\|_{t',q+t',\iota}^- + C'_N \left\|\alpha\right\|_{p-1,q+1,\iota} \right) \\
&\leq \bar{\varrho}^N (\sigma^N/2) \left\|\alpha\right\|'_{p,q,\iota} + C'_N \sum_{t'=0}^p \frac{K^{-t'-1}}{1-K^{-1}} \left\|\alpha\right\|_{t',q+t',\iota}^- + \frac{C'_N}{1-K^{-1}} \left\|\alpha\right\|_{p-1,q+1,\iota} \\
&\leq \bar{\varrho}^N (\sigma^N/2) \left\|\alpha\right\|'_{p,q,\iota} + \frac{C'_N K^{-1}}{1-K^{-1}} \left\|\alpha\right\|'_{p,q,\iota} + 2C'_N \left\|\alpha\right\|_{p-1,q+1,\iota}.
\end{aligned}$$

Since K was chosen large enough, we have therefore

$$\left\|\mathcal{L}_{\pi,\phi}^N \alpha\right\|'_{p,q,\iota} \leq \bar{\varrho}^N \sigma^N \left\|\alpha\right\|'_{p,q,\iota} + 2C'_N \left\|\alpha\right\|_{p-1,q+1,\iota}. \tag{4.5}$$

By the inductive assumption, the iterates of $\mathcal{L}_{\pi,\phi}$ satisfy the inequality

$$\left\|\mathcal{L}_{\pi,\phi}^n \alpha\right\|_{p-1,q+1,\iota} \leq C_1 n^d \bar{\varrho}^n \left\|\alpha\right\|_{p-1,q+1,\iota}, \tag{4.6}$$

for some constant C_1. This implies by induction on m that

$$\begin{aligned}\left\|\mathcal{L}_{\pi,\phi}^{mN}\alpha\right\|_{p,q,\iota}' &\leq (\bar\varrho\sigma)^{mN}\left\|\alpha\right\|_{p,q,\iota}' + 2C_N'\sum_{k=1}^{m}(\bar\varrho\sigma)^{(k-1)N}\left\|\mathcal{L}_{\pi,\phi}^{(m-k)N}\alpha\right\|_{p-1,q+1,\iota}\\ &\leq (\bar\varrho\sigma)^{mN}\left\|\alpha\right\|_{p,q,\iota}' + 2C_N' C_1 (mN)^d \bar\varrho^{mN}\bar\varrho^{-N}\left(\sum_{i=0}^{\infty}\sigma^{iN}\right)\left\|\alpha\right\|_{p-1,q+1,\iota}.\end{aligned}$$

Finally, taking care of the first N iterates, we obtain:

$$\left\|\mathcal{L}_{\pi,\phi}^{n}\alpha\right\|_{p,q,\iota}' \leq C\bar\varrho^n\sigma^n\left\|\alpha\right\|_{p,q,\iota}' + Cn^d\bar\varrho^n\left\|\alpha\right\|_{p-1,q+1,\iota}. \tag{4.7}$$

Since the norms $\left\|\cdot\right\|_{p,q,\iota}$ and $\left\|\cdot\right\|_{p,q,\iota}'$ are equivalent, this concludes the proof. □

4.1.2. *Compact embedding of $\mathcal{B}^{p,q,\iota}$ in $\mathcal{B}^{p-1,q+1,\iota}$.*

Lemma 4.2. *Assume that (t,q,ι) is correct and that $(t+1,q-1,\iota)$ is also correct. There exists a constant $C>0$ such that, for all $\varepsilon>0$, for all W, W' which are (C_Σ,ε)-close,*[11] *for all $\alpha\in\mathcal{S}$,*

$$\sup_{\omega'=(W',\varphi',v_1',\dots,v_t')\in\Omega_{t,q+t,\iota}}|\ell_{\omega'}(\alpha)| \leq C\sup_{\omega=(W,\varphi,v_1,\dots,v_t)\in\Omega_{t,q+t,\iota}}|\ell_\omega(\alpha)| + C\varepsilon\left\|\alpha\right\|_{t+1,q+t,\iota}^{-}.$$

Proof. Let v be a vector field with $|v|_{\mathcal{C}^{r-1}}\leq\varepsilon$ whose flow ϕ_u satisfies $\phi_1(W)=W'$ and $W^u=\phi_u(W)\in\Sigma$ for $0\leq u\leq 1$, and is bounded in $\mathcal{C}^r$ by C_Σ. Start from $\omega'=(W',\varphi',v_1',\dots,v_t')\in\Omega_{t,q+t,\iota}$. Define vector fields $v_i^u=\phi_{1-u}^* v_i'$, $v^u=\phi_{1-u}^* v$ and functions $\varphi^u=\varphi'\circ\phi_{1-u}$. Let

$$F(u)=\int_W \varphi^0 L_{v_1^0}\dots L_{v_t^0}(\phi_u^*\alpha)=\int_{W^u}\varphi^u L_{v_1^u}\dots L_{v_t^u}(\alpha). \tag{4.8}$$

Then $F(1)=\ell_{\omega'}(\alpha)$, and $F(0)=\int_W\varphi^0 L_{v_1^0}\dots L_{v_t^0}(\alpha)$. Since the vector fields v_i^0 have a uniformly bounded $\mathcal{C}^{q+t-\iota}$ norm, and φ^0 is uniformly bounded in $\mathcal{C}^{q+t}$, it is sufficient to prove that $|F(1)-F(0)|\leq C\varepsilon\left\|\alpha\right\|_{t+1,q+t,\iota}^{-}$ to conclude. We will prove such an estimate for $F'(u)$.

We have

$$F'(u)=\int_W\varphi^0 L_{v_1^0}\dots L_{v_t^0}(\phi_u^* L_v\alpha)=\int_{W^u}\varphi^u L_{v_1^u}\dots L_{v_t^u}L_{v^u}\alpha. \tag{4.9}$$

By definition of $\left\|\cdot\right\|_{t+1,q+t,\iota}^{-}$, this quantity is bounded by $C\left\|\alpha\right\|_{t+1,q+t,\iota}^{-}$, which concludes the proof. □

Assume that (p,q,ι) is correct and $p>0$. Hence, $(p-1,q+1,\iota)$ is also correct. Moreover, for any $\alpha\in\mathcal{S}$, $\left\|\alpha\right\|_{p-1,q+1,\iota}\leq\left\|\alpha\right\|_{p,q,\iota}$. Hence, there exists a canonical map $\mathcal{B}^{p,q,\iota}\to\mathcal{B}^{p-1,q+1,\iota}$ extending the identity on the dense subset $\mathcal{S}$ of $\mathcal{B}^{p,q,\iota}$.

Lemma 4.3. *If (p,q,ι) is correct and $p>0$, the canonical map from $\mathcal{B}^{p,q,\iota}$ to $\mathcal{B}^{p-1,q+1,\iota}$ is compact.*

Proof. The main point of the proof of Lemma 4.3 is to be able to work only with a finite number of leaves. This is ensured by Lemma 4.2. The rest of the proof is then very similar to [GL06, Proof of Lemma 2.1]. □

[11] See Definition 3.1 for the definition of (C_Σ,ε)-close.

4.1.3. *Spectral gap.* Lemmas 4.1 and 4.3, giving a Lasota-Yorke inequality and compactness, imply a precise spectral description of the transfer operator $\mathcal{L}_{\pi,\phi}$. Let

$$\varrho := \limsup_{n\to\infty} \varrho_n. \tag{4.10}$$

Proposition 4.4. *Assume that (p,q,ι) is correct. The operator $\mathcal{L}_{\pi,\phi} : \mathcal{S} \to \mathcal{S}$ extends to a continuous operator on $\mathcal{B}^{p,q,\iota}$. Its spectral radius is at most ϱ and its essential spectral radius is at most* $\max(\lambda^{-p}, \nu^q)\varrho$.

Proof. For any $\bar\varrho > \varrho$, the inequality (4.3), the compactness Lemma 4.3 and Hennion's Theorem [Hen93] prove that the spectral radius of $\mathcal{L}_{\pi,\phi}$ acting on $\mathcal{B}^{p,q,\iota}$ is bounded by $\bar\varrho$, and that its essential spectral radius is bounded by $\max(\lambda^{-p}, \nu^q)\bar\varrho$. Letting $\bar\varrho$ tend to ϱ, we obtain the required upper bounds on the spectral radius and essential spectral radius of $\mathcal{L}_{\pi,\phi}$. □

4.2. **A lower bound for the spectral radius.** We will prove that the spectral radius of $\mathcal{L}_{\pi,\phi}$ is in fact *equal* to ϱ. To do this, we will need the following lower bound on ϱ_n. Since we will use this lemma again later, to exclude the possibility of Jordan blocks, we formulate it in greater generality than currently needed.

Lemma 4.5. *Assume that (p,q,ι) is correct. Let α be an element of $\mathcal{B}^{p,q,\iota}$ which induces a nonnegative measure on every admissible leaf $W \in \Sigma$. Assume moreover that there exists an open set O containing Λ such that, for any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that, for any $x \in O \cap \bigcap_{n\geq 0} T^n V$, for any $W \in \Sigma$ containing x with* $\operatorname{dist}(x, \partial W) > \varepsilon$, *holds* $\int_{B_W(x,\varepsilon)} \alpha \geq c_\varepsilon$.[12] *Then there exist $L \in \mathbb{N}$ and $C > 0$ such that, for all large enough n,*

$$\varrho_n^n \leq C \left\| \mathcal{L}_{\pi,\phi}^{n-2L} \alpha \right\|_{p,q,\iota}. \tag{4.11}$$

Proof. Let $W \in \Sigma$, and let W_j be a covering of $T^{-n}W^{(n)}$ as given by Definition 3.1. All is needed is to prove the inequality

$$\sum_j |e^{S_n\phi}\pi_n|_{\mathcal{C}^{r-1+\iota}(W_j)} \leq C \left\| \mathcal{L}_{\pi,\phi}^{n-2L} \alpha \right\|_{p,q,\iota}. \tag{4.12}$$

The lemma would have a two lines proof if we could use distortion to estimate $|e^{S_n\phi}\pi_n|_{\mathcal{C}^{r-1+\iota}(W_j)}$ by $\int_{W_j} e^{S_n\phi}\pi_n\alpha$, but there are two problems in doing so. First, π vanishes at some points, hence classical distortion controls do not apply. Second, the behavior of α is known only for leaves close to Λ. To overcome these two problems, we will consider a small neighborhood of Λ, where π_n is equal to 1 and α is well behaved. We can assume without loss of generality that $\pi = 1$ on O.

Recall the definition of the constant δ_0 in the first item of Definition 3.1. Decreasing δ_0 if necessary, we can assume that, for all $x \in \Lambda$, $B(x, 3\delta_0) \subset O$. Then there exist $\varepsilon > 0$ and a small neighborhood O' of Λ with the following property: *let $x \in O'$, and let Z be a submanifold of dimension d_s containing x, whose tangent space is everywhere contained in the stable cone, and with* $\operatorname{dist}(x, \partial Z) \geq \delta_0$. *Then there exists a point $y \in Z \cap O \cap \bigcap_{n\geq 0} T^n V$ such that* $\operatorname{dist}(y, \partial Z) \geq \varepsilon$ *and* $\operatorname{dist}(x, y) \leq \delta_0$. This is a consequence of the compactness of Λ and the uniform transversality between the stable cones and the unstable leaves. Decreasing O' if necessary, we can assume that

$$\forall x \in O', \quad B(x, 2\delta_0) \subset O. \tag{4.13}$$

[12]Here, $B_W(x,\varepsilon)$ denotes the ball of center x and radius ε in the manifold W.

We can also assume $\varepsilon < \delta_0$.

We will use the following fact: *there exists $L \in \mathbb{N}$ such that, for any point x, for any $n \geq 2L$, if $T^i x \in V$ for all $0 \leq i \leq n-1$ then $T^i x \in O'$ for all $L \leq i \leq n-L$.*

This is a classical property of locally maximal sets, proved as follows. If the fact were not true, we would have for all $L \geq 0$ a point $x_L \in V \backslash O'$ such that $T^i x_L \in V$ for all $|i| \leq L$. An accumulation point of the sequence x_L would then belong to $\overline{V} \backslash O'$, and also to $\bigcap_{n \in \mathbb{Z}} T^{-n} U$. This is a contradiction since this last intersection is equal to Λ by assumption, and is therefore contained in O'.

Let us now return to the proof. We start from the covering $\{W_j\}$ of $T^{-n} W^{(n)}$. Fix some j such that π_n is not zero on W_j. There exists $x_j \in W_j$ such that $T^i x_j \in V$ for $0 \leq i \leq n-1$. The above fact ensures that $T^i x_j \in O'$ for $L \leq i \leq n-L$. By definition of the enlargement W^e of W, the point $T^n x_j$ belongs to $\{y \in W^e : \operatorname{dist}(y, \partial W^e) \geq \delta_0\}$. Since T^{-1} expands the distances in the stable cone, we get $\operatorname{dist}(T^L x_j, \partial(T^{-(n-L)} W^e)) \geq \delta_0$. Therefore, the above property shows the existence of a point $y_j \in T^{-(n-L)} W^e \cap O \cap \bigcap_{n \geq 0} T^n V$, with $\operatorname{dist}(T^L x_j, y_j) \leq \delta_0$, such that the ball B_j of center y_j and radius ε in the manifold $T^{-(n-L)} W^e$ is well defined. This ball satisfies $\int_{B_j} \alpha \geq c_\varepsilon$ by the assumption of the lemma. Moreover, by contraction of the iterates of T along $T^{-(n-L)} W^e$, we have $T^i(B_j) \subset B(T^{L+i} x_j, 2\delta_0)$ for $0 \leq i \leq n-L$. Since $T^{L+i} x_j \in O'$ for $0 \leq i \leq n-2L$, (4.13) shows that $T^i(B_j) \subset O$ for $0 \leq i \leq n-2L$. Therefore, $\pi_{n-2L} = 1$ on B_j.

By uniform contraction of T, $|\pi_n|_{\mathcal{C}^r(W_j)} \leq C$. Moreover, usual distortion estimates show that $|e^{S_n \phi}|_{\mathcal{C}^{r-1+\iota}(W_j)} \leq C |e^{S_n \phi}|_{\mathcal{C}^0(W_j)}$, and that $|e^{S_{n-2L} \phi}|_{\mathcal{C}^0(T^L W_j)} \leq C \inf_{x \in B_j} e^{S_{n-2L} \phi(x)}$. Using these estimates, we can compute:

$$\begin{aligned}
|e^{S_n \phi} \pi_n|_{\mathcal{C}^{r-1+\iota}(W_j)} &\leq C |e^{S_n \phi}|_{\mathcal{C}^0(W_j)} \leq C |e^{S_{n-2L} \phi}|_{\mathcal{C}^0(T^L W_j)} \\
&\leq C |e^{S_{n-2L} \phi}|_{\mathcal{C}^0(T^L W_j)} \int_{B_j} \alpha \leq C \int_{B_j} e^{S_{n-2L} \phi} \alpha \\
&= C \int_{B_j} e^{S_{n-2L} \phi} \pi_{n-2L} \alpha = C \int_{T^{n-2L} B_j} \mathcal{L}_{\pi,\phi}^{n-2L} \alpha.
\end{aligned}$$

Summing over j and using the fact that there is a bounded number of overlap,

$$\sum_j |e^{S_n \phi} \pi_n|_{\mathcal{C}^{r-1+\iota}(W_j)} \leq C \int_{O \cap T^{-L} W^e} \mathcal{L}_{\pi,\phi}^{n-2L} \alpha. \tag{4.14}$$

Since the set of integration can be covered by a uniformly bounded number of admissible leaves, we get

$$\sum_j |e^{S_n \phi} \pi_n|_{\mathcal{C}^{r-1+\iota}(W_j)} \leq C \left\| \mathcal{L}_{\pi,\phi}^{n-2L} \alpha \right\|_{p,q,\iota}. \tag{4.15}$$

□

Corollary 4.6. *Assume that (p, q, ι) is correct. The spectral radius of $\mathcal{L}_{\pi,\phi}$ acting on $\mathcal{B}^{p,q,\iota}$ is exactly ϱ.*

Proof. Choose once and for all an element α_r of $\mathcal{S}$ induced by a Riemannian metric, as explained in Remark 2.2. It satisfies the assumptions of Lemma 4.5. Therefore, for some constants $L > 0$ and $C > 0$,

$$\varrho_n^n \leq C \left\| \mathcal{L}_{\pi,\phi}^{n-2L} \alpha_r \right\|_{p,q,\iota} \leq C \left\| \mathcal{L}_{\pi,\phi}^{n-2L} \right\|_{p,q,\iota}. \tag{4.16}$$

Letting n tend to infinity, we obtain that the spectral radius of $\mathcal{L}_{\pi,\phi}$ is at least $\limsup \varrho_n = \varrho$. The result follows remembering Proposition 4.4. □

4.3. **First description of the peripheral eigenvalues.** In this paragraph, we will study the eigenvalues of modulus ϱ. The main goal is to prove that the eigenfunctions for eigenvalues of modulus ϱ are in fact measures. Fix a correct (p, q, ι).

Denote by $(\gamma_i \varrho)_{i=1}^M$ the peripheral eigenvalues of $\mathcal{L}_{\pi,\phi}$ acting on $\mathcal{B}^{p,q,\iota}$, with $|\gamma_i| = 1$. Let κ be the size of the largest Jordan block. Since $\mathcal{L}_{\pi,\phi} : \mathcal{B}^{p,q,\iota} \to \mathcal{B}^{p,q,\iota}$ is quasicompact, it must have the form

$$\mathcal{L}_{\pi,\phi} = \sum_{i=1}^{M} (\gamma_i \varrho S_{\gamma_i} + N_{\gamma_i}) + R \tag{4.17}$$

where $S_{\gamma_i}, N_{\gamma_i}$ are finite rank operators such that $S_{\gamma_i} S_{\gamma_j} = \delta_{ij} S_{\gamma_i}$, $S_{\gamma_i} N_{\gamma_j} = N_{\gamma_j} S_{\gamma_i} = \delta_{ij} N_{\gamma_j}$, $N_{\gamma_i} N_{\gamma_j} = \delta_{ij} N_{\gamma_i}^2$, $S_{\gamma_i} R = R S_{\gamma_i} = N_{\gamma_i} R = R N_{\gamma_i} = 0$, $N_{\gamma_i}^\kappa = 0$, and R has spectral radius strictly smaller than ϱ. Accordingly, for each $|\gamma| = 1$, holds

$$\lim_{n\to\infty} n^{-\kappa} \sum_{k=0}^{n-1} \gamma^{-k} \varrho^{-k} \mathcal{L}_{\pi,\phi}^k = \frac{1}{\kappa!} \sum_{i=1}^{M} N_{\gamma_i}^{\kappa-1} \delta_{\gamma\gamma_i}. \tag{4.18}$$

In this formula, if $\kappa = 1$, then $N_{\gamma_i}^{\kappa-1}$ indicates the eigenprojection corresponding to the eigenvalue $\gamma_i \varrho$, i.e., S_{γ_i}. We will denote by F_{γ_i} the image of $N_{\gamma_i}^{\kappa-1}$.

Lemma 4.7. *There exists $C > 0$ such that, for all $n > 0$,*

$$\varrho_n^n \le C n^{\kappa-1} \varrho^n. \tag{4.19}$$

This lemma implies in particular that we can apply Lemma 4.1 to $(\bar{\varrho}, d) = (\rho, \kappa - 1)$.

Proof. There exists a constant $C > 0$ such that $\left\| \mathcal{L}_{\pi,\phi}^n \right\|_{p,q,\iota} \le C n^{\kappa-1} \varrho^n$. Equation (4.16) then implies $\varrho_n^n \le C n^{\kappa-1} \varrho^n$. □

Lemma 4.8. *For all γ with $|\gamma| = 1$, and all $\alpha \in F_\gamma$, there exists $C > 0$ such that, for all $W \in \Sigma$, for all $t \le p$, for all $v_1, \dots, v_t \in \mathcal{V}^{q+t-\iota}(W)$ with $|v_i|_{\mathcal{C}^{q+t-\iota}} \le 1$, for all $\varphi \in \mathcal{C}_0^{q+t}(W)$,*

$$\left| \int_W \varphi \cdot L_{v_1} \dots L_{v_t} \alpha \right| \le C |\varphi|_{\mathcal{C}^t(W)}. \tag{4.20}$$

The point of this lemma is that the upper bound depends only on $|\varphi|_{\mathcal{C}^t}$ while the naive upper bound would use $|\varphi|_{\mathcal{C}^{q+t}(W)}$.

Proof. We can apply Lemma 4.1 (and more precisely the inequality (4.2)) to $(\bar{\varrho}, d) = (\varrho, \kappa - 1)$, and to the parameters $(t, 0, \iota)$. We get

$$\left\| \mathcal{L}_{\pi,\phi}^n \right\|_{t,0,\iota} \le C n^{\kappa-1} \varrho^n. \tag{4.21}$$

Since $\mathcal{S}$ is dense in $\mathcal{B}^{p,q,\iota}$, we have $N_\gamma^{\kappa-1} \mathcal{B}^{p,q,\iota} = N_\gamma^{\kappa-1} \mathcal{S}$. Therefore, we can write α as $N_\gamma^{\kappa-1}(\tilde{\alpha})$ where $\tilde{\alpha} \in \mathcal{S}$. Then, by (4.18),

$$\int_W \varphi \cdot L_{v_1} \dots L_{v_t} \alpha = \lim_{n\to\infty} \frac{\kappa!}{n^\kappa} \sum_{k=0}^{n-1} (\gamma\varrho)^{-k} \int_W \varphi \cdot L_{v_1} \dots L_{v_t} (\mathcal{L}_{\pi,\phi}^k \tilde{\alpha}). \tag{4.22}$$

Moreover, these integrals satisfy

$$\left|\int_W \varphi \cdot L_{v_1} \dots L_{v_t}(\mathcal{L}^k_{\pi,\phi}\tilde{\alpha})\right| \leq |\varphi|_{C^t(W)} \left\|\mathcal{L}^k_{\pi,\phi}\tilde{\alpha}\right\|_{t,0,\iota}, \tag{4.23}$$

by definition of $\|\cdot\|_{t,0,\iota}$ (this last norm is well defined since $\tilde{\alpha} \in \mathcal{S}$). Using the inequality (4.21) and the last two equations, we get the lemma. □

Choose α_r as in Corollary 4.6 and let $\alpha_0 := \frac{1}{\kappa!}N_1^{\kappa-1}\alpha_r$. Clearly $\mathcal{L}_{\pi,\phi}\alpha_0 = \varrho\alpha_0$.

Lemma 4.9. *Assume that (p, q, ι) is correct and $p > 0$. Take γ with $|\gamma| = 1$, and $\alpha \in F_\gamma$, Then, for each $W \in \Sigma$, α defines a measure on W. In addition, all such measures are absolutely continuous, with bounded density, with respect to the one induced by α_0.*

Proof. For $t = 0$, Lemma 4.8 shows that $\left|\int_W \varphi\alpha\right| \leq C|\varphi|_{C^0}$. This shows that α induces a measure on each $W \in \Sigma$.

For $\gamma = 1$ and $\tilde{\alpha} = \alpha_r$, $t = 0$, Equation (4.22) shows that α_0 is a nonnegative measure. Moreover, whenever $\varphi \in \mathcal{C}^q(W)$, it also implies

$$\left|\int_W \varphi\alpha\right| \leq C\int_W |\varphi|\alpha_0.$$

This inequality extends to continuous functions by density. Hence, the measure defined by α is absolutely continuous with respect to the one defined by α_0 (with bounded density). □

An element α of F_γ defines a measure on each element of Σ. Moreover, if W and W' intersect, and $\varphi \in \mathcal{C}^q$ is supported in their intersection, then $\int_W \varphi\alpha = \int_{W'} \varphi\alpha$. Indeed, this is the case for any element of $\mathcal{B}^{p,q,\iota}$, since it holds trivially for an element of $\mathcal{S}$, and $\mathcal{S}$ is dense in $\mathcal{B}^{p,q,\iota}$. Therefore, the measures on elements of Σ defined by an element of F_γ match locally, and can be glued together: if an oriented submanifold of dimension d_s is covered by elements of Σ, then an element of F_γ induces a measure on this submanifold. We will denote by $\mathcal{M}\alpha$ the measure induced by α on each oriented stable leaf in U.

Lemma 4.10. *The map $\alpha \mapsto \mathcal{M}\alpha$ is injective on each set F_γ. Moreover, $\alpha_0 \neq 0$.*

Proof. Let $\alpha \in F_\gamma$ satisfy $\mathcal{M}\alpha = 0$, we will first prove that

$$\|\alpha\|_{0,q,\iota} = 0. \tag{4.24}$$

Notice first that Lemma 4.2 shows that, if $W' \in \Sigma$ is (C_Σ, ε)-close to an element W of Σ contained in a stable manifold, then

$$\left|\int_{W'} \varphi\alpha\right| \leq C\varepsilon|\varphi|_{\mathcal{C}^q(W')}. \tag{4.25}$$

Indeed, the assumption $\mathcal{M}\alpha = 0$ shows that, for any $\varphi \in \mathcal{C}^q_0(W)$, $\ell_{(W,\varphi)}(\alpha) = 0$.

Take now $W \in \Sigma$ and $\varphi \in \mathcal{C}^q_0(W)$. Using the partition of unity on $T^{-n}W^{(n)}$ given by the definition of admissible leaves, we get

$$\int_W \varphi\alpha = \int_W \varphi(\gamma\rho)^{-n}\mathcal{L}^n_{\pi,\phi}\alpha = (\gamma\rho)^{-n}\sum_{j=1}^{k}\int_{W_j} \varphi \circ T^n \rho_j \pi_n e^{S_n\phi} \cdot \alpha. \tag{4.26}$$

Each W_j is $(C_\Sigma, \varepsilon_n)$-close to an element of Σ contained in a stable leaf, where $\varepsilon_n \to 0$ is given by the definition of admissible sets of leaves. Hence, (4.25) shows that $\left|\int_W \varphi\alpha\right|$ is bounded by

$$\varrho^{-n} \sum_{j=1}^{k} C\varepsilon_n |\pi_n e^{S_n\phi}|_{\mathcal{C}^q(W_j)} \leq C\varrho^{-n}\varrho_n^n \varepsilon_n. \tag{4.27}$$

The sequence $\varrho^{-n}\varrho_n^n$ grows at most subexponentially, while ε_n goes exponentially fast to 0 by Definition 3.1. Therefore, this quantity goes to 0, hence (4.24).

Next, if $\alpha \in F_\gamma$, then $\mathcal{L}_{\pi,\phi}^n \alpha = (\gamma\varrho)^n \alpha$. Using the Lasota-Yorke inequality (4.4) (applied to $(\rho, \kappa - 1)$ by Lemma 4.7), we get for some $\sigma < 1$

$$\|\alpha\|_{p,q,\iota} = \varrho^{-n} \left\|\mathcal{L}_{\pi,\phi}^n \alpha\right\|_{p,q,\iota} \leq C\sigma^n \|\alpha\|_{p,q,\iota}, \tag{4.28}$$

since $\|\alpha\|_{0,q+p,\iota} \leq \|\alpha\|_{0,q,\iota} = 0$ by (4.24). Choosing n large yields $\|\alpha\|_{p,q,\iota} = 0$.

Let us now prove $\alpha_0 \neq 0$. Otherwise, $\mathcal{M}\alpha_0 = 0$. For any $\alpha \in F_\gamma$, the measure $\mathcal{M}\alpha$ is absolutely continuous with respect to $\mathcal{M}\alpha_0$, hence zero. By injectivity of the map $\alpha \mapsto \mathcal{M}\alpha$, we get $\alpha = 0$. Therefore, there is no eigenfunction corresponding to an eigenvalue of modulus ϱ. This contradicts Corollary 4.6. $\square$

5. Peripheral Spectrum and Topology

In this section we establish a connection between the peripheral spectrum of the operator and the topological properties of the dynamical systems at hand.

5.1. **Topological description of the dynamics.** Let us recall the classical *spectral decomposition* of a map T as above (see e.g. [HK95, Theorem 18.3.1]). Assume that $T : U \to X$ is a diffeomorphism and that $\Lambda = \bigcap_{n\in\mathbb{Z}} T^n U$ is a compact locally maximal hyperbolic set. Then there exist disjoint closed sets $\Lambda_1, \dots, \Lambda_m$ and a permutation σ of $\{1, \dots, m\}$ such that $\bigcup_{i=1}^m \Lambda_i = NW(T_{\restriction\Lambda})$, the nonwandering set of the restriction of T to Λ. Moreover, $T(\Lambda_i) = \Lambda_{\sigma(i)}$, and when $\sigma^k(i) = i$ then $T^k_{\restriction\Lambda_i}$ is topologically mixing, and Λ_i is a compact locally maximal hyperbolic set for T^k.

Hence, to understand the dynamics of T on Λ (and especially its invariant measures) when $\Lambda = NW(T_{\restriction\Lambda})$, it is sufficient to understand the case when $T_{\restriction\Lambda}$ is topologically mixing.

To deal with orientation problems, we will in fact need more than mixing. Let $\bar\Lambda$ be the set of pairs (x, E) where $x \in \Lambda$ and $E \in \mathcal{G}$ is $E^s(x)$ with one of its two possible orientations. Let $\bar T : \bar\Lambda \to \bar\Lambda$ be the map induced by DT on $\bar\Lambda$, and let $\mathrm{pr} : \bar\Lambda \to \Lambda$ be the canonical projection. We have a commutative diagram

$$\begin{tikzcd}
\bar\Lambda \arrow[r, "\bar T"] \arrow[d, "\mathrm{pr}"'] & \bar\Lambda \arrow[d, "\mathrm{pr}"] \\
\Lambda \arrow[r, "T"] & \Lambda
\end{tikzcd}$$

Moreover, the fibers of pr have cardinal exactly 2. When T is topologically mixing, there are exactly three possibilities:

- Either $\bar T$ is also topologically mixing. In this case, we say that T is *orientation mixing.*

- Or there is a decomposition $\bar\Lambda = \bar\Lambda_1 \cup \bar\Lambda_2$ where each $\bar\Lambda_i$ is invariant under $\bar T$, and the restriction of pr to each $\bar\Lambda_i$ is an isomorphism. We say that T is *mixing, but orientation preserving.*
- Or there is a decomposition $\bar\Lambda = \bar\Lambda_1 \cup \bar\Lambda_2$ such that pr is an isomorphism on each $\bar\Lambda_i$, and $\bar T$ exchanges $\bar\Lambda_1$ and $\bar\Lambda_2$. In this case, T^2 is orientation preserving as defined before.

To understand the spectral properties of T, it is sufficient to understand the first two cases, since the last one can be reduced to the second one by considering T^2.

In the second case, there exists an orientation of the spaces $E^s(x)$ for $x \in \Lambda$, which depends continuously on x, and is invariant under DT. Let us say arbitrarily that this orientation is positive. Consequently, if the neighborhood U of Λ is small enough, there exists a decomposition of $\{(x,E) \;:\; x \in U, E \in \mathcal{G} \text{ with } E \subset C_s(x)\}$ into two disjoint sets S_+ and S_-, the first one corresponding to vector spaces E whose orientation is close to the positive orientation of a nearby set $E^s(x)$, and the other one corresponding to the opposite orientation. The sets S_+ and S_- are invariant under the action of DT. Let $\mathcal{B}^{p,q,\iota}_\pm$ denote the closure in $\mathcal{B}^{p,q,\iota}$ of the elements of $\mathcal{S}$ which vanish on $S_\mp$. Then

$$\mathcal{B}^{p,q,\iota} = \mathcal{B}^{p,q,\iota}_+ \oplus \mathcal{B}^{p,q,\iota}_-.$$

The transfer operator $\mathcal{L}_{\pi,\phi}$ leaves invariant the sets $\mathcal{B}^{p,q,\iota}_+$ and $\mathcal{B}^{p,q,\iota}_-$. Moreover, there is a natural isomorphism from $\mathcal{B}^{p,q,\iota}_+$ to $\mathcal{B}^{p,q,\iota}_-$ (corresponding to reversing the orientation), which conjugates the action of $\mathcal{L}_{\pi,\phi}$ on $\mathcal{B}^{p,q,\iota}_+$ and $\mathcal{B}^{p,q,\iota}_-$. Hence, the spectral data of $\mathcal{L}_{\pi,\phi}$ acting on $\mathcal{B}^{p,q,\iota}$ are simply twice the corresponding data for the corresponding action on $\mathcal{B}^{p,q,\iota}_+$. Therefore, when T is mixing but orientation preserving, we can restrict ourselves to the study of $\mathcal{L}_{\pi,\phi}$ acting on $\mathcal{B}^{p,q,\iota}_+$.

5.2. **The peripheral spectrum in the topologically mixing case.** In this paragraph we will assume that the dynamics has no wandering parts, that is $NW(T_{\restriction\Lambda}) = \Lambda$. Given the discussion of the previous section we can thus restrict ourselves to the mixing case. Under such an assumption we obtain a complete characterization of the peripheral spectrum. Note that the proof of the next theorem relies on some general properties of conformal leafwise measures that, for the reader's convenience, are proved in Section 9.

Theorem 5.1. *Assume that T is orientation mixing (respectively mixing but orientation preserving). Consider the operator $\mathcal{L}_{\pi,\phi}$ acting on $\mathcal{B}^{p,q,\iota}$ (resp. $\mathcal{B}^{p,q,\iota}_+$). Then ϱ is a simple eigenvalue, and there is no other eigenvalue of modulus ϱ.*

Proof. We give the proof e.g. for the orientation mixing case, the other one is analogous.

Let us first prove that $\kappa = 1$, that is, there is no Jordan block. We will show that α_0 satisfies the assumptions of Lemma 4.5. Assume on the contrary that there exists a small ball B on which the integral of α_0 vanishes, centered at a point of $\bigcap_{n\in\mathbb{N}} T^n V$. The preimages of such a small ball accumulate on the stable manifolds of T. By invariance, the integral of α_0 still vanishes on $T^{-n}B$. Taking a subsequence and passing to the limit, we obtain a small ball B' in a stable manifold, centered at a point of Λ, on which $\alpha_0 = 0$. There is a point x in $\Lambda \cap B'$ such that $\{T^{-n}x\}$ is dense in Λ. Let $\varepsilon > 0$ be such that the measure $\mathcal{M}\alpha_0$ induced by α_0 (as defined in Paragraph 4.3) vanishes on $B(x,\varepsilon)$. Using the invariance of α_0 and the expansion properties of T^{-n}, this implies that $\mathcal{M}\alpha_0 = 0$ on each ball $B(T^{-n}x, \varepsilon)$. By continuity and density, $\mathcal{M}\alpha_0 = 0$. This is in contradiction with Lemma 4.10.

Therefore, we can apply Lemma 4.5 to α_0, and get $\varrho_n^n \leq C \left\| \mathcal{L}_{\pi,\phi}^{n-2L} \alpha_0 \right\|_{p,q,\iota}$. Since α_0 is an eigenfunction for the eigenvalue ϱ, this yields $\varrho_n^n \leq C\varrho^n$. The Lasota-Yorke inequality (4.2) yields $\left\| \mathcal{L}_{\pi,\phi}^n \right\|_{p,q,\iota} \leq C\varrho^n$. Hence, there can be no Jordan block.

Let us now prove that α_0 is the only eigenfunction (up to scalar multiplication) corresponding to an eigenvalue of modulus ϱ. Let α be such an eigenfunction, for an eigenvalue $\gamma\varrho$, with $|\gamma| = 1$ and $\alpha \neq 0$. Notice first that the leafwise measure $\mathcal{M}\alpha$ is a continuous leafwise measure, in the sense of Section 9. Indeed, if the test function φ is $\mathcal{C}^q$, then the continuity property of leafwise measures is clear for any element of $\mathcal{S}$, and extends by density to any element of $\mathcal{B}^{p,q,\iota}$. When $\alpha \in F_\gamma$, this continuity property extends from $\mathcal{C}^q$ test functions to $\mathcal{C}^0$ test functions by Lemma 4.9. Let us check the assumptions of Proposition 9.4 (for the map T^{-1}). Note first that T^{-1} is topologically mixing on Λ by assumption, and expanding along stable leaves. Moreover, let U be an open set in a stable leaf, containing a point $x \in \Lambda$. Since T^{-1} is transitive, there exists a nearby point y whose orbit under T^{-1} is dense in Λ. The point $z = [x, y] = W^s(x) \cap W^u(y)$ belongs to $\Lambda \cap U$ if y is close enough to x, and its orbit under T^{-1} is also dense in Λ. Hence, Proposition 9.4 applies, and shows that the measure $\mathcal{M}\alpha$ is proportional to $\mathcal{M}\alpha_0$. Since $\mathcal{M}\alpha \neq 0$ by Lemma 4.10, it follows that $\gamma = 1$. Moreover, the equality $\mathcal{M}\alpha = \gamma'\mathcal{M}\alpha_0$ implies $\alpha = \gamma'\alpha_0$, again by Lemma 4.10. □

In the course of the above proof, we have showed that α_0 gives a positive mass to each ball in a stable manifold, centered at a point of Λ. By compactness of Λ and the continuity properties of α_0, this implies the following useful fact:

For any $\delta > 0$, there exists $c_\delta > 0$ such that, for any ball $B(x, \delta)$ in the stable manifold of a point $x \in \Lambda$,

$$\int_{B(x,\delta)} \alpha_0 \geq c_\delta. \tag{5.1}$$

Remark 5.2. *For the case of unilateral subshifts of finite type, or more generally when the transfer operator acts on spaces of continuous functions, there is a much simpler argument to exclude the existence of Jordan blocks (see* [Kel89] *or* [Bal00]*), which goes as follows.*

Assume that the spectral radius of $\mathcal{L}$ is ϱ, and that there exists an eigenfunction $g > 0$ corresponding to this eigenvalue. Then, for any function f, there exists $C > 0$ such that $|f| \leq Cg$. Therefore, if the size κ of the corresponding Jordan block is > 1,

$$\frac{1}{n^\kappa} \left| \sum_{k=0}^{n-1} \varrho^{-k} \mathcal{L}^k f \right| \leq C \frac{1}{n^\kappa} \sum_{k=0}^{n-1} \varrho^{-k} \mathcal{L}^k g \to 0. \tag{5.2}$$

Hence, $\frac{1}{n^\kappa} \sum_{k=0}^{n-1} \varrho^{-k} \mathcal{L}^k f$ converges to 0 in the C^0 norm. But it converges to the eigenprojection of f in the strong norm, so this eigenprojection has to be 0 for all f. This is a contradiction, and $\kappa = 1$.

Unfortunately, this simple argument does not apply in our setting since the elements of our spaces are not functions: even if we have constructed the analogue of the function g, i.e., α_0, there is no such inequality as $|\alpha| \leq C\alpha_0$ for a general $\alpha \in \mathcal{S}$. This explains why we had to resort to a more sophisticated proof.

6. Invariant measures and the variational principle

6.1. **Description of the invariant measure.** In this paragraph, we assume that T is a map on a compact locally maximal hyperbolic set, which is either orientation mixing, or mixing but orientation preserving. Choose $p \in \mathbb{N}^*$ and $q > 0$ such that (p, q, ι) is correct. In the first case, we let $\mathcal{B} = \mathcal{B}^{p,q,\iota}$ and in the second case $\mathcal{B} = \mathcal{B}^{p,q,\iota}_+$. The transfer operator $\mathcal{L}_{\pi,\phi}$ acts on $\mathcal{B}$ and has a simple eigenvalue at ϱ and no other eigenvalue of modulus ϱ, by Theorem 5.1.

Let α_0 be the eigenfunction of ϱ. The dual operator acting on $\mathcal{B}'$ also has a simple eigenvalue at ϱ. Let ℓ_0 denote the corresponding eigenfunction, normalized so that $\ell_0(\alpha_0) = 1$.

Lemma 6.1. *There exists a constant $C > 0$ such that, for all $\varphi \in \mathcal{C}^r(U)$, $|\ell_0(\varphi\alpha_0)| \le C|\varphi|_{\mathcal{C}^0}$. Moreover, $\ell_0(\varphi\alpha_0) = \ell_0(\varphi \circ T \cdot \alpha_0)$.*

Proof. Let us show that, for any $\alpha \in \mathcal{B}$,

$$|\ell_0(\alpha)| \le C \|\alpha\|_{0,p+q,\iota} . \tag{6.1}$$

Since $\ell_0 = \varrho^{-n}\mathcal{L}'^{\,n}_{\pi,\phi}\ell_0$,

$$\begin{aligned}|\ell_0(\alpha)| &= \varrho^{-n}|\ell_0(\mathcal{L}^n_{\pi,\phi}\alpha)| \le C\varrho^{-n} \left\|\mathcal{L}^n_{\pi,\phi}\alpha\right\|_{p,q,\iota} \\ &\le C\varrho^{-n}\left[C\sigma^n\varrho^n \|\alpha\|_{p,q,\iota} + C\varrho^n\|\alpha\|_{0,p+q,\iota}\right]\end{aligned}$$

for some $\sigma < 1$, by (4.4). Letting n tend to ∞, we obtain (6.1).

Lemma 4.9 for $t = 0$ implies that $\|\varphi\alpha_0\|_{0,p+q,\iota} \le C|\varphi|_{\mathcal{C}^0}$. Together with (6.1), this leads to $|\ell_0(\varphi\alpha_0)| \le C|\varphi|_{\mathcal{C}^0}$.

Finally, we have

$$\begin{aligned}\ell_0(\varphi\alpha_0) = (\varrho^{-1}\mathcal{L}'_{\pi,\varphi}\ell_0)(\varphi\alpha_0) &= \varrho^{-1}\ell_0(\mathcal{L}_{\pi,\varphi}(\varphi\alpha_0)) \\ &= \ell_0(\varphi\circ T^{-1}\cdot\varrho^{-1}\mathcal{L}_{\pi,\phi}\alpha_0) = \ell_0(\varphi\circ T^{-1}\cdot\alpha_0).\end{aligned}$$

This proves the last assertion of the lemma. □

Lemma 6.1 shows that the functional

$$\mu : \varphi \mapsto \ell_0(\varphi\alpha_0),$$

initially defined on $\mathcal{C}^r$ functions, extends to a continuous functional on continuous functions. Hence, it is given by a (complex) measure, that we will also denote by μ. Lemma 6.1 also shows that this measure is invariant. Hence, it is supported on the maximal invariant set in U, i.e., Λ.

Lemma 6.2. *The measure μ is a (positive) probability measure.*

Proof. By equation (4.18), the subsequent definition of α_0 and Theorem 5.1 it follows that, for each $\alpha \in \mathcal{B}^{p,q,\iota}$,

$$\lim_{n\to\infty} \varrho^{-n}\mathcal{L}^n_{\pi,\phi}\alpha = \ell_0(\alpha)\alpha_0 \tag{6.2}$$

with $\ell_0(\alpha_r) = 1$. Hence, for all $\varphi_1, \varphi_2 \ge 0$ and $W \in \Sigma$ holds

$$0 \le \lim_{n\to\infty}\int_W \varphi_1\varrho^{-n}\mathcal{L}^n_{\pi,\phi}(\varphi_2\alpha_r) = \ell_0(\varphi_2\alpha_r)\int_W \varphi_1\alpha_0. \tag{6.3}$$

We know that the measure defined by α_0 is nonnegative, and nonzero by Lemma 4.10. Therefore, there exist W and φ_1 such that $\int_W \varphi_1\alpha_0 > 0$. We get, for any $\varphi_2 \geq 0$, $\ell_0(\varphi_2\alpha_r) \geq 0$. If $\varphi \geq 0$, we have (since ℓ_0 is an eigenfunction of $\mathcal{L}'_{\pi,\varphi}$)

$$\begin{aligned}\ell_0(\varphi\alpha_0) &= \lim_{n\to\infty} \ell_0(\varphi\varrho^{-n}\mathcal{L}^n_{\pi,\phi}\alpha_r) = \lim_{n\to\infty} \ell_0(\varrho^{-n}\mathcal{L}^n_{\pi,\phi}(\varphi\circ T^n\alpha_r)) \\ &= \lim_{n\to\infty} \ell_0(\varphi\circ T^n\alpha_r) \geq 0.\end{aligned}$$

Hence, the measure μ is positive. The normalization $\ell_0(\alpha_0) = 1$ ensures that it is a probability measure. □

Using the spectral information on $\mathcal{L}$, we can now prove the characterization of the correlations for the measure μ stated in Theorem 1.2. This concludes the proof of Theorem 1.2 provided one shows that μ is indeed the unique Gibbs measure, this will be done in Theorem 6.4.

Proof of Theorem 1.2. We will first describe an abstract setting which implies the conclusion of the theorem, and then show that hyperbolic maps fit into this setting.

Let T be a map on a space X, preserving a probability measure μ. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two spaces of functions on X. Assume that there exist a Banach space $\mathcal{B}$, a continuous linear operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ and two continuous maps $\Phi_1 : \mathcal{F}_1 \to \mathcal{B}$ and $\Phi_2 : \mathcal{F}_2 \to \mathcal{B}$ such that, for all $n \in \mathbb{N}$, for all $\psi_1 \in \mathcal{F}_1$ and $\psi_2 \in \mathcal{F}_2$,

$$\int \psi_1 \cdot \psi_2 \circ T^n \,\mathrm{d}\mu = \langle \Phi_2(\psi_2), \mathcal{L}^n\Phi_1(\psi_1)\rangle. \tag{6.4}$$

Then, for any σ strictly larger than the essential spectral radius of $\mathcal{L}$, there exist a finite dimensional space F, a linear map M on F, and two continuous maps $\tau_1 : \mathcal{F}_1 \to F$ and $\tau_2 : \mathcal{F}_2 \to F'$ such that (1.2) holds. This is indeed a direct consequence of the spectral decomposition of the operator $\mathcal{L}$.

In our specific setting, we take for $\mathcal{B}$ the Banach space defined above, $\mathcal{L} = \varrho^{-1}\mathcal{L}_{\pi,\phi}$, $\mathcal{F}_1$ is the closure of the set of $\mathcal{C}^r$ functions in $\mathcal{C}^p(U)$ and $\mathcal{F}_2$ is the closure of the set of $\mathcal{C}^r$ functions in $\mathcal{C}^q(U)$. On the set of $\mathcal{C}^r$ functions, define $\Phi_1(\psi_1) = \psi_1\alpha_0$, and $\Phi_2(\psi_2) = \psi_2\ell_0$. By construction, (6.4) holds. We have to check that Φ_1 and Φ_2 can be continuously extended respectively to $\mathcal{F}_1$ and $\mathcal{F}_2$. Let us first prove

$$\|\psi\alpha_0\|_{p,q,\iota} \leq C|\psi|_{\mathcal{C}^p}. \tag{6.5}$$

This will imply that Φ_1 can be extended by continuity to $\mathcal{F}_1$.

To check (6.5), consider $t \leq p$, let $W \in \Sigma$, let $v_1, \dots, v_t \in \mathcal{V}^{q+t-\iota}(W)$ and let $\varphi \in \mathcal{C}_0^{q+t}(W)$. Then

$$\int_W \varphi \cdot L_{v_1}\dots L_{v_t}(\psi\alpha_0) = \sum_{A\subset\{1,\dots,t\}} \int_W \varphi \left(\prod_{i\in A} L_{v_i}\right)\psi \cdot \left(\prod_{i\notin A} L_{v_i}\right)\alpha_0. \tag{6.6}$$

Using Lemma 4.8 to bound each of these integrals, we get an upper bound of the form $C|\psi|_{\mathcal{C}^p}$. This proves (6.5).

Let us now extend Φ_2. By (6.1), for any $\alpha \in \mathcal{B}$,

$$|\Phi_2(\psi)(\alpha)| \leq C\|\psi\alpha\|_{0,p+q,\iota} \leq C\|\psi\alpha\|_{0,q,\iota} \leq C|\psi|_{\mathcal{C}^q}\|\alpha\|_{0,q,\iota} \leq C|\psi|_{\mathcal{C}^q}\|\alpha\|_{p,q,\iota}.$$

Hence, $\|\Phi_2(\psi)\| \leq C|\psi|_{\mathcal{C}^q}$. In particular, Φ_2 can be continuously extended to $\mathcal{F}_2$.

The proof is almost complete, there is just a technical subtlety to deal with. Since p is an integer, $\mathcal{F}_1 = \mathcal{C}^p(U)$. However, when q is not an integer, $\mathcal{C}^r(U)$ is not dense in $\mathcal{C}^q(U)$, hence $\mathcal{F}_2$ is strictly included in $\mathcal{C}^q(U)$. To bypass this technical problem, we rather use $q' < q$ close enough to q so that $\sigma > \max(\lambda^{-p}, \nu^{q'})$ (where σ is the precision up to which we want a description of the correlations, as in the statement of the theorem). Let $\mathcal{F}_2$ be the closure of $\mathcal{C}^r(U)$ in $\mathcal{C}^{q'}(U)$. For $\psi_1 \in \mathcal{F}_1$ and $\psi_2 \in \mathcal{F}_2$, we get as above a description of the correlations, with an error term at most $C\sigma^n |\psi_1|_{\mathcal{C}^p(U)} |\psi_2|_{\mathcal{C}^{q'}(U)}$. Since $\mathcal{F}_2$ contains $\mathcal{C}^q(U)$, and $|\psi_2|_{\mathcal{C}^{q'}(U)} \leq |\psi_2|_{\mathcal{C}^q(U)}$, this gives the required upper bound for all functions of $\mathcal{C}^q(U)$. □

6.2. **Variational principle.** We will denote by $B_n(x,\varepsilon)$ the dynamical ball of length n for T^{-1}, i.e.,

$$B_n(x,\varepsilon) = \{y \in U \ : \ \forall\, 0 \leq i \leq n-1,\ d(T^{-i}y, T^{-i}x) \leq \varepsilon\}.$$

Proposition 6.3. *For all small enough $\varepsilon > 0$, there exist constants $A_\varepsilon, a_\varepsilon > 0$ such that, for all $n \in \mathbb{N}$ and all $x \in \Lambda$,*

$$a_\varepsilon e^{S_n\bar\phi(T^{-n}x)} \varrho^{-n} \leq \mu(B_n(x,\varepsilon)) \leq \mu(\overline{B_n(x,\varepsilon)}) \leq A_\varepsilon e^{S_n\bar\phi(T^{-n}x)} \varrho^{-n}, \tag{6.7}$$

where $\bar\phi$ is defined by $\bar\phi(y) = \phi(y, E^s(y))$.

Proof. Let φ be a nonnegative $\mathcal{C}^r$ function supported in $B_n(x,\varepsilon)$, bounded by one, and equal to one on $B_n(x,\varepsilon/2)$. We will prove

$$a_\varepsilon e^{S_n\bar\phi(T^{-n}x)} \varrho^{-n} \leq \mu(\varphi) \leq A_\varepsilon e^{S_n\bar\phi(T^{-n}x)} \varrho^{-n}, \tag{6.8}$$

which will conclude the proof.

Let $W \in \Sigma$, and let $\varphi_0 \in \mathcal{C}^q_0(W)$ with $|\varphi_0|_{\mathcal{C}^q(W)} \leq 1$. Then

$$\begin{aligned} \int_W \varphi_0 \varphi \alpha_0 &= \int_W \varphi_0 \varphi \varrho^{-n} \mathcal{L}^n_{\pi,\phi} \alpha_0 \\ &= \sum_j \int_{W_j} \rho_j \varphi_0 \circ T^n \varphi \circ T^n \varrho^{-n} e^{S_n\phi} \pi_n \cdot \alpha_0, \end{aligned} \tag{6.9}$$

where ρ_j is the partition of unity on $T^{-n}W^{(n)}$ given by the definition of admissible leaves. Since φ is supported in $B_n(x,\varepsilon)$, the number of leaves W_j on which $\varphi \circ T^n$ is nonzero is uniformly bounded. On each of these leaves, $e^{S_n\phi}$ is bounded by $Ce^{S_n\bar\phi(T^{-n}x)}$. It follows that

$$\left| \int_W \varphi_0 \varphi \alpha_0 \right| \leq C \varrho^{-n} e^{S_n\bar\phi(T^{-n}x)}.$$

Since this estimate is uniform in W and φ_0, the upper bound is proven.

For the (trickier) lower bound, we proceed in four steps.

First step. Let us show that, for any piece W of stable leaf containing a point y with $d(x,y) < \varepsilon/10$ and $\operatorname{dist}(y, \partial W) \geq 10\varepsilon$*, we have*

$$\int_W \varphi \alpha_0 \geq C_\varepsilon \varrho^{-n} e^{S_n\bar\phi(T^{-n}x)}. \tag{6.10}$$

Indeed, $T^{-n}W$ contains a disk D centered at a point of Λ, of radius $\varepsilon/10$, and contained in $T^{-n}B_n(x,\varepsilon/2)$. The integral of α_0 on such a disk is uniformly bounded from below by a constant C_ε (by (5.1)), and $\varphi \circ T^n = 1$ on D. Therefore,

$$\int_W \varphi\alpha_0 = \int_W \varphi \varrho^{-n} \mathcal{L}^n_{\pi,\phi}\alpha_0 = \varrho^{-n} \int_{T^{-n}W} \varphi \circ T^n e^{S_n\phi} \pi_n \alpha_0 \geq \varrho^{-n} \int_D e^{S_n\phi}\pi_n\alpha_0.$$

Moreover, $\pi_n\alpha_0 = \alpha_0$ on D by (9.1), and $e^{S_n\phi} \geq Ce^{S_n\bar\phi(T^{-n}x)}$ on D. This proves (6.10).

Second step. Let us show that, for any $\delta > 0$, there exists $M = M(\varepsilon, \delta)$ such that, for any $m \geq M$, there exists $C = C(\varepsilon, \delta, m)$ such that, for any piece W of stable manifold containing a point $y \in \Lambda$ with $\operatorname{dist}(y, \partial W) \geq \delta$,

$$\int_{T^{-m}W} \varphi\alpha_0 \geq C\varrho^{-n} e^{S_n\bar{\phi}(T^{-n}x)}. \tag{6.11}$$

This is a direct consequence of the topological mixing of T on Λ: if m is large enough, then $T^{-m}W$ will contain a subset W' satisfying the assumptions of the first step. Therefore, (6.10) implies the conclusion.

Third step. Let $W \in \Sigma$ be a piece of stable manifold containing a point of Λ in its interior. Denote by W^e its enlargement, as in Definition 3.1. There exists $C = C(\varepsilon, W) > 0$ such that, for any large enough $p \in \mathbb{N}$,

$$\int_{W^e} \varrho^{-p}\mathcal{L}^p_{\pi,\phi}(\varphi\alpha_0) \geq C\varrho^{-n} e^{S_n\bar{\phi}(T^{-n}x)}. \tag{6.12}$$

To prove this, consider $\{W_j\}$ a covering of $T^{-p}W^{(p)}$ as in the definition of admissible leaves, and ρ_j the corresponding partition of unity.

As in the proof of Lemma 4.5, there exists an integer L with the following property: to each W_j, we can associate a small ball $B(y_j, \delta)$ contained in $T^{-(p-L)}W^e$, at a bounded distance from $T^L W_j$, with $y_j \in \Lambda$. Increasing L if necessary (this process does not decrease δ), we can assume $L \geq M(\varepsilon, \delta)$. Since the balls B_j have a bounded number of overlaps,

$$\int_{W^e} \mathcal{L}^p_{\pi,\phi}(\varphi\alpha_0) \geq C \sum_j \int_{B_j} \pi_{p-L} e^{S_{p-L}\phi} \mathcal{L}^L_{\pi,\phi}(\varphi\alpha_0). \tag{6.13}$$

The function π_{p-L} is equal to 1 on a neighborhood of the support of α_0, by (9.1), so we can disregard it. Moreover, $\inf_{B_j} e^{S_{p-L}\phi} \geq Ce^{S_{p-L}(y_j)}$. We get

$$\int_{W^e} \mathcal{L}^p_{\pi,\phi}(\varphi\alpha_0) \geq C \sum_j e^{S_{p-L}\bar{\phi}(y_j)} \int_{T^{-L}B_j} e^{S_L\phi}\varphi\alpha_0. \tag{6.14}$$

The second step applies to each of the sets B_j. Since $e^{S_L\phi}$ is uniformly bounded from below, we obtain

$$\begin{aligned}\varrho^n \int_{W^e} \mathcal{L}^p_{\pi,\phi}(\varphi\alpha_0) &\geq Ce^{S_n\bar{\phi}(T^{-n}x)} \sum_j e^{S_{p-L}\bar{\phi}(y_j)} \geq Ce^{S_n\bar{\phi}(T^{-n}x)} \sum_j \int_{T^L W_j} e^{S_{p-L}\phi}\alpha_0 \\ &\geq Ce^{S_n\bar{\phi}(T^{-n}x)} \int_W \mathcal{L}^{p-L}_{\pi,\phi}\alpha_0 = Ce^{S_n\bar{\phi}(T^{-n}x)} \int_W \varrho^{p-L}\alpha_0,\end{aligned}$$

since α_0 is an eigenfunction of $\mathcal{L}_{\pi,\phi}$.

Fourth Step. Conclusion. Fix $W \in \Sigma$ satisfying the assumptions of the third step. When $p \to \infty$, $\varrho^{-p}\mathcal{L}^p_{\pi,\phi}(\varphi\alpha_0)$ converges to $\ell_0(\varphi\alpha_0)\alpha_0 = \mu(\varphi)\alpha_0$. Passing to the limit in (6.12), we obtain

$$\mu(\varphi) \int_{W^e} \alpha_0 \geq C\varrho^{-n} e^{S_n\bar{\phi}(T^{-n}x)}. \tag{6.15}$$

This is the desired lower bound. $\square$

Theorem 6.4. *The spectral radius ϱ is equal to the topological pressure $e^{P_{\text{top}}(\bar{\phi})}$ of the function $\bar{\phi}$. In addition, the measure μ is the unique probability measure satisfying the variational principle*

$$h_\mu(T) + \int \bar{\phi}\, \mathrm{d}\mu = P_{\text{top}}(\bar{\phi}).$$

In other words, μ is the so-called *Gibbs measure* of $T : \Lambda \to \Lambda$, corresponding to the potential $\bar{\phi}$.

Proof. The theorem is a completely general consequence of Lemma 6.3. Indeed, let T be any continuous transformation on a compact space Λ preserving an ergodic probability measure μ. Let $\bar{\phi}$ be a function such that Lemma 6.3 is satisfied, and there exists $C > 0$ such that, for any dynamical ball $B = B_n(x, \varepsilon)$, $\sup_B e^{S_n\bar{\phi}} \leq C \inf_B e^{S_n\bar{\phi}}$ (which is satisfied in our hyperbolic setting since $\bar{\phi}$ is Hölder continuous). Then μ satisfies the variational principle and is the unique measure to do so. This result is due to Bowen, and is proved e.g. in [HK95, Theorem 20.3.7]. For the convenience of the reader, let us sketch the proof.

Recall that the definition of the topological pressure of $\bar{\phi}$ is given by

$$P_{\text{top}}(\bar{\phi}) := \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \ln S_d(T, \bar{\phi}, \varepsilon, n) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \ln N_d(T, \bar{\phi}, \varepsilon, n)$$

where

$$S_d(T, \bar{\phi}, \varepsilon, n) := \inf \left\{ \sum_{x\in E} e^{S_n\bar{\phi}(T^{-n}x)} \;:\; \Lambda \subset \bigcup_{x\in E} B_n(x, \varepsilon) \right\}$$

$$N_d(T, \bar{\phi}, \varepsilon, n) := \sup \left\{ \sum_{x\in E} e^{S_n\bar{\phi}(T^{-n}x)} \;:\; E \subset \Lambda \text{ is } (n, \varepsilon)\text{-separated} \right\}.$$

Now in the first case

$$1 = \mu(\Lambda) \leq \sum_{x\in E} \mu(B_n(x, \varepsilon)) \leq A_\varepsilon \varrho^{-n} \sum_{x\in E} e^{S_n\bar{\phi}(T^{-n}x)}$$

Taking the inf on E and the limits yields $\varrho \leq P_{\text{top}}(\bar{\phi})$. On the other hand if E is (n, ε)-separated, holds

$$1 = \mu(\Lambda) \geq \sum_{x\in E} \mu(B_n(x, \varepsilon/2)) \geq a_{\varepsilon/2} \varrho^{-n} \sum_{x\in E} e^{S_n\bar{\phi}(T^{-n}x)}$$

which, taking the sup on E and the limits, yields $\varrho \geq P_{\text{top}}(\bar{\phi})$.

Finally, if ν is any invariant ergodic probability measure, the Brin-Katok local entropy theorem [BK83] states that the quantity

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log(1/\nu(B_n(x, \varepsilon)))$$

converges ν almost everywhere to $h_\nu(T)$. Lemma 6.3 shows that, μ-a.e.,

$$P_{\text{top}}(\bar{\phi}) - \limsup_{n\to\infty} \frac{S_n\bar{\phi}(T^{-n}x)}{n} \geq h_\mu(T) \geq P_{\text{top}}(\bar{\phi}) - \liminf_{n\to\infty} \frac{S_n\bar{\phi}(T^{-n}x)}{n}.$$

By Birkhoff Theorem, for μ-almost all x, $\frac{S_n\bar{\phi}(T^{-n}x)}{n}$ converges to $\int \bar{\phi}\, \mathrm{d}\mu$. Together with the above inequalities, we get

$$h_\mu(T) + \int \bar{\phi}\, \mathrm{d}\mu = P_{\text{top}}(\bar{\phi}).$$

Hence μ maximizes the variational principle. To show that the maximizing probability is unique one can proceed exactly as in [HK95, Theorem 20.3.7] where one uses Lemma 6.3 instead of [HK95, Lemma 20.3.4]. □

Remark 6.5. *Theorem 6.4 implies in particular that the measure μ constructed using the transfer operator $\mathcal{L}_{\pi,\phi}$ is in fact independent of the truncation π. This can also be checked directly by spectral arguments. However, α_0 and ℓ_0* do *depend on the truncation: if we take a truncation with smaller support π', such that $\pi = 1$ on the support of π', then the new eigenfunctions α_0' and ℓ_0' are equal to $\alpha_0 \cdot \prod_{i=1}^{N} \pi' \circ T^{-i}$ and $\ell_0 \cdot \prod_{i=0}^{N-1} \pi' \circ T^i$ for any large enough N. Nevertheless, this shows that they coincide with α_0 and ℓ_0 on a neighborhood of Λ.*

7. Relationships with the classical theory of Gibbs measures

7.0.1. *Margulis' construction.* Classically, the Gibbs measure can be constructed by coding, but there is also a geometric construction, due initially to Margulis. He proves the following result (for the measure of maximal entropy in [Mar04], but the proofs extend to Gibbs measures, see e.g. [BL98]):

There exist a family of measures μ^s on the stable leaves, supported on Λ, and a family of measures μ^u on unstable leaves, supported on Λ, such that

$$\mu^s = T_*(e^{\bar\phi - P_{\mathrm{top}}(\bar\phi)}\mu^s), \quad \mu^u = T_*(e^{P_{\mathrm{top}}(\bar\phi)-\bar\phi}\mu^u). \tag{7.1}$$

The measures μ^s are constructed by starting from the Riemannian measure on a very large piece of stable leaf, and then pushing it by the dynamics T^n (with a suitable multiplication by the weight $e^{\bar\phi}$). The sequence is shown to converge in some sense, to the invariant set of measures μ^s. This corresponds exactly to what we do by the iteration of the transfer operator, exhibiting α_0 as the limit of $\mathcal{L}_{\pi,\phi}^n(\alpha_r)$. The main difference is that we get the convergence in a strong sense (norm convergence), and for free due to the spectral properties of the operator. In fact, the measures μ^s are exactly the measures induced by α_0 on the stable leaves.

The measures μ^u are constructed in the same way, but iterating T^{-1}. The relationship with our abstract eigenfunction ℓ_0 in the dual of $\mathcal{B}$ is less clear at first sight. However, they are still very closely related. Indeed, let us define an element $\ell \in \mathcal{B}'$ as follows: if $\alpha \in \mathcal{B}$, and φ is a $\mathcal{C}^r$ function supported in a small open set foliated by small stable leaves, and having as transversal a small unstable leaf F, set

$$\ell(\varphi\alpha) = \int_{x\in F}\left(\int_{y\in W^s(x)} \varphi(y)\prod_{k=0}^{\infty}\pi\circ T^k(y)e^{\sum_{k=0}^{\infty}\bar\phi(T^k y)-\bar\phi(T^k x)}\alpha\right)\mathrm{d}\mu_F^u(x). \tag{7.2}$$

This is well defined since the function $y \mapsto \prod_{k=0}^{\infty}\pi\circ T^k(y)e^{\sum_{k=0}^{\infty}\bar\phi(T^k y)-\bar\phi(T^k x)}$ is $\mathcal{C}^{r-1+\iota}$ on each stable leaf (the product is in fact finite, since $\pi\circ T^k$ is uniformly equal to 1 for large enough k), and can therefore be integrated against α. The Jacobian of the holonomy of the stable foliation with respect to the measures μ^u is exactly $e^{\sum_{k=0}^{\infty}\bar\phi(T^k y)-\bar\phi(T^k x)}$. Hence, the local definition of ℓ is independent of the choice of the transversal F. Using a partition of unity $\varphi_1,\dots,\varphi_n$, we have a well defined element $\ell\in\mathcal{B}'$.

The conformality property of the measures μ^u implies that $\mathcal{L}'_{\pi,\phi}\ell = \varrho\ell$. Indeed, let us compute locally:

$$\ell(\mathcal{L}_{\pi,\phi}\alpha) = \int_{x\in F}\left(\int_{y\in W^s(x)}\prod_{k=0}^{\infty}\pi\circ T^k(y)e^{\sum_{k=0}^{\infty}\bar\phi(T^k y)-\bar\phi(T^k x)}\mathcal{L}_{\pi,\phi}\alpha\right)\mathrm{d}\mu^u_F(x)$$

$$= \int_{x'\in T^{-1}F}\left(\int_{y'\in W^s(x')}\prod_{k=1}^{\infty}\pi\circ T^k(y')e^{\sum_{k=1}^{\infty}\bar\phi(T^k y')-\bar\phi(T^k x')}\pi(y')e^{\bar\phi(y')}\alpha\right)\mathrm{d}\mu^u_F(x).$$

The equality $\mu^u = T_*(e^{P_{\mathrm{top}}(\bar\phi)-\bar\phi}\mu^u)$ gives $\mathrm{d}\mu^u_F(x) = e^{P_{\mathrm{top}}(\bar\phi)-\bar\phi(x')}\,\mathrm{d}\mu^u_{T^{-1}F}(x')$. It follows that $\ell(\mathcal{L}_{\pi,\phi}\alpha) = \varrho\ell(\alpha)$.

Since the eigenspace of $\mathcal{L}'_{\pi,\phi}$ is one-dimensional, this shows that ℓ and ℓ_0 are proportional. Hence, the measures μ^u give a geometric description of ℓ_0.

Remark 7.1. *This description implies that*

$$|\ell_0(\psi\alpha)| \le C\,\|\alpha\|_{0,q,\iota}\cdot\sup_{x\in\Lambda}|\psi|_{\mathcal{C}^q(W^s(x))}.$$

Hence, in (1.2)*, the factor* $|\psi|_{\mathcal{C}^q(U)}$ *can be replaced with* $\sup_{x\in\Lambda}|\psi|_{\mathcal{C}^q(W^s(x))}$.

Finally, the Gibbs measure μ is constructed by "putting together locally" the measures μ^s and μ^u. In our setting, this task is automatically performed by the functional analytic framework.

7.0.2. *Currents.* Another classical construction of Gibbs measures, closely related to the previous one but expressed slightly differently, is to work with *currents*, [RS75]. A current of degree k is an element of the dual of the space of smooth differential forms of degree $d-k$, where d is the dimension of the ambient manifold (which we shall assume to be oriented in this paragraph). A differential form of degree k gives a current of degree k, since it is possible to take its exterior product against a form of degree $d-k$, and then integrate on the whole manifold.

A way to construct Gibbs measures is to find "conformal currents" in the stable and unstable directions (i.e., currents satisfying a condition similar to (7.1)), and then take their "intersection" to get an invariant measure, which is the Gibbs measure.

Since the differential forms of degree d_s form a subset of $\mathcal{B}$ (see Remark 2.1), an element of the dual of $\mathcal{B}$ gives rise to a current of degree d_u. In particular, the eigenfunction ℓ_0 is a current (and (7.2) shows that it is even a current with an interesting underlying geometric structure). Hence, ℓ_0 can be interpreted as a conformal current in the unstable direction.

On the other hand, α_0 is not a current of dimension d_s in a natural way. Indeed, there is no canonical way to multiply an element of $\mathcal{S}$ with a differential form to get something which could be integrated. However, assume that the weight ϕ belongs to $\mathcal{W}^1$ (i.e., it depends only on the point), and that T is mixing but orientation preserving. Then we can consider in $\mathcal{B}$ the closure $\mathcal{C}$ of the set of differential forms. An element of $\mathcal{C}$ is naturally a current.[13] Since $\phi\in\mathcal{W}^1$, it is easy to check that $\mathcal{L}_{\pi,\phi}$ leaves $\mathcal{C}$ invariant. Moreover, the spectral radius of the restriction of $\mathcal{L}_{\pi,\phi}$ to $\mathcal{C}$ is still ϱ (notice that this would *not* hold in the orientation mixing case). This implies that

[13]To see this we must check that, if α is a smooth form of degree d_u, there exists $C>0$ such that, for any form β of degree d_s, $|\int\alpha\wedge\beta|\le C\,\|\beta\|_{\mathcal{B}}$. This can be checked in coordinates by using a basis of the tangent space whose elements all belong to the stable cone.

the eigenfunction α_0 belongs to $\mathcal{C}$, hence α_0 can then be interpreted as a current. Finally, μ is indeed constructed by "intersecting" the two conformal currents ℓ_0 and α_0 (this intersection process, which is often complicated to implement in general, is given here for free by the functional analytic framework).

7.0.3. *Young-Chernov-Dolgopyat.* In recent years a new approach has been introduced by Lai-Sang Young. It has been further simplified by Dolgopyat and then Dolgopyat-Chernov and has been recently reviewed in [Che06]. Such an approach is indeed very close to the one described here. Essentially, it uses objects in the dual of our spaces $\mathcal{B}^{0,q}$.

More precisely, $\Omega_{p,q,\iota}$, $p+q < r-1+\iota$, can be endowed with a topology τ, stronger than the weak-* one, for which it is compact.[14] This implies an interesting characterization of the dual spaces of $\mathcal{B} := \mathcal{B}^{p,q,\iota}$.

Lemma 7.2. *Let $\ell_* \in \mathcal{B}'$, then there exists a Borel (with respect to the τ topology) measure ρ on Ω such that, for all $h \in \mathcal{B}$,*

$$\ell_*(h) = \int_\Omega \ell(h)\ \rho(d\ell).$$

Proof. The first step is to construct $F : \mathcal{B} \to \mathcal{C}^0(\Omega, \mathbb{C})$ defined by

$$F(h)(\ell) := \ell(h),$$

since τ is stronger than the weak-* topology, $F(h)$ is continuous. Call $A := F(\mathcal{B})$, clearly A is a closed linear space in $\mathcal{C}^0(\Omega, \mathbb{C})$. We can then associate to ℓ_* the element $\nu \in A'$ defined by $\nu(F(h)) = \ell_*(h)$. By the Hahn-Banach Theorem there exists an extension ν' of ν to all $\mathcal{C}^0(\Omega, \mathbb{C})$. A this point, by the Riesz representation Theorem, there exists a measure ρ on Ω such that

$$\nu'(f) = \int_\Omega f(\ell)\rho(d\ell).$$

Hence, for each $h \in \mathcal{B}$, we have

$$\ell_*(h) = \nu(F(h)) = \nu'(F(h)) = \int_\Omega F(h)(\ell)\rho(d\ell) = \int_\Omega \ell(h)\rho(d\ell). \qquad \square$$

Accordingly, the elements $(W, \varphi) \in \Omega_{0,q}$ correspond exactly to the *standard pairs* in [Che06] and, by the above Lemma, the basic objects used in [Che06] are precisely the elements of $(\mathcal{B}^{0,q})'$.

The difference lies in the technique used to prove statistical properties: in [Che06] is used a probabilistic coupling technique (instead of the functional analytic one) to prove statistical properties. Such an approach yields much weaker results than the present one but it needs much less structure and hence it is amenable to generalizations in the non-uniformly hyperbolic case.

7.0.4. *Gouëzel-Liverani.* In [GL06], we introduced an approach to study the SRB measure of an Anosov map. In many respects, it has the same flavor as the approach in the present paper, with admissible leaves and norms obtained in a very similar way. There are however two important differences between the two papers.

- On the technical level, the proof of the Lasota-Yorke inequality (4.3) was more complicated since we had not realized one could use weighted norms.

[14] Essentially, two manifolds are close if they are $\mathcal{C}^{r-\varepsilon}$ close, for $p+q+\varepsilon < r-1+\iota$, and the φ must be $\mathcal{C}^{p+q-\varepsilon}$ close and the vector fields $\mathcal{C}^{p+q-\iota-\varepsilon}$ close.

- More conceptually, we had not distinguished between what is specific to the SRB measure and comes from the Riemannian setting, and what is completely general. In particular, we considered our spaces $\mathcal{B}^{p,q}$ as spaces of distributions, by integrating in the transverse direction with respect to Lebesgue measure. This is very natural in this case since Lebesgue measure is precisely the transverse measure μ^u of Margulis, i.e., the eigenelement ℓ_0 in the dual space is already given for free at the beginning. However, this is really a peculiarity of the SRB measure, that we had to avoid to treat general Gibbs measures. This explains why we get spaces of generalized differential forms instead of spaces of distributions.

8. Examples and Applications

In this section we try to give an idea of the breadth of the results by first discussing some natural examples to which it can be applied and then illustrating an interesting consequence: perturbation theory.

8.1. **Examples.**

8.1.1. *Anosov and Axiom-A.* Clearly the theory applies to any Anosov or, more generally, Axiom-A system. In particular, it allows to construct and investigate the SRB measures and the measures of maximal entropy. In this respect the present work contains an alternative, self contained, construction yielding the classical results contained in [Bow75].[15] The relation between the present approach and other, more classical, ones are discussed in some detail in Section 7.

8.1.2. *Open systems.* Systems of physical interest are often open, that is the particles can leave the system. This can happen either with certainty, once they enter in a given region (holes), or according to some probability distribution π (holes in noisy systems). The first case cannot be treated in the present setting since the boundaries of the hole introduce discontinuities in the system but the latter can be treated provided π is smooth. For example, consider an Anosov system (X, T) and the following dynamics: a point disappears with probability $\pi(x)dx$ and then, if it has not disappeared, it is mapped by T. In this situation a typical quantity of physical interest is the escape rate with respect to Lebesgue, that is the rate at which mass leaks out of the system. If ϕ is the potential corresponding to the SRB measure, then the transfer operator associated to the above dynamics is simply $\mathcal{L}_{\phi,1-\pi}$ and the escape rate is nothing else than the logarithm of its leading eigenvalue.

8.1.3. *Billiards with no eclipse conditions.* An interesting concrete system to which the present paper applies is the scattering by convex obstacle with no-eclipse condition (that is the convex hull of any two scatterers does not intersect any other scatter). Although the reflection from an obstacle gives rise to singularities in the Poincaré section, nevertheless the no-eclipse condition implies that only points that will leave the system can experience a tangent collision (corresponding to a singularity), hence there exists a neighborhood of the set of the points that keep being scattered forever in which the dynamics is smooth, hence falls in our setting. See [KS97] for a pleasant introduction to such a subject. In particular, one can obtain

[15] Notice however that we have an additional smoothness assumption on the weight.

sharper information on the spectrum of the Ruelle operator that are available by the usual coding techniques used in [Mor91, Sto01, Mor04].

8.2. **An application: smoothness with respect to parameters.** As already mentioned, the present setting easily allows to discuss the dependence from parameters of various physically relevant quantities.

Let us make a simple example to illustrate such a possibility. Let (X, T_λ) be a one parameter family of Anosov maps and let ϕ_λ be a one parameter family of potentials. Suppose that T_λ, ϕ_λ are jointly $\mathcal{C}^r$ in the variable and the parameter. By applying the perturbation theory in [GL06, Section 8] it follows that the leading eigenvalue and the corresponding eigenmeasure are smooth in λ. If, for example, we are interested in the measure of maximal entropy ($\phi_\lambda = 0$ in view of the variational principle given in Theorem 6.4), then it follows that, for any $\varepsilon > 0$, the topological entropy $h_\lambda = P_{\mathrm{top}}(0, T_\lambda)$ is $\mathcal{C}^{\lfloor r \rfloor - 1 - \varepsilon}$ (this is obvious, since this quantity is constant!) and the measure of maximal entropy μ_λ is a $\mathcal{C}^{\lfloor r \rfloor - 1 - \varepsilon}$ function of λ as a function from $\mathbb{R}$ to $\mathcal{D}'_r$ (that is, if viewed as a distribution of order r).

In fact, the formalism makes it possible to easily compute the derivatives of the various objects involved. We illustrate this possibility with the following proposition. Write T_λ as $I_\lambda \circ T_0$ where I_λ is the flow from time 0 to time λ of a $\mathcal{C}^{r-1}$ time dependent vector field v_t. If v is a smooth vector field, denote by v^s and v^u its projections on the stable and unstable bundles (they are only Hölder continuous vector fields), and by L_v its Lie derivative. If Φ is a smooth function on $\mathcal{G}$ such that $\Phi(E)$ is independent of the orientation of E, let $\bar{\Phi}(x) = \Phi(x, E^s(x))$. The formula (7.2) for ℓ_0 shows that, for such a Φ,

$$\ell_0(\Phi \alpha_0) = \mu_0(\bar{\Phi}). \tag{8.1}$$

Proposition 8.1. *Let $A = \bar{\phi}'_0 - \sum_{n=0}^{\infty} L_{v_0^s}(\bar{\phi}_0 \circ T_0^n)$. Then $h'_0 = \mu_0(A)$ and, if φ is a $\mathcal{C}^1$ test function,*

$$\left.\frac{d\mu_\lambda(\varphi)}{d\lambda}\right|_{\lambda=0} = \sum_{k=-\infty}^{\infty} \mu_0(\varphi \circ T_0^k (A - h'_0)) + \sum_{k=-\infty}^{-1} \mu_0(L_{v_0^u}(\varphi \circ T_0^k)) - \sum_{k=0}^{\infty} \mu_0(L_{v_0^s}(\varphi \circ T_0^k)).$$

Notice that the sums in this last equation are clearly finite (the different terms decay to 0 exponentially fast). Notice also that, when the potential ϕ_λ is constant, we get $h'_0 = 0$ and, in the same way, $h'_\lambda = 0$. This proves that the topological entropy is locally constant, without using as usual the structural stability of the map.

Proof. Due to (8.1), we can omit the bars everywhere and work only with ϕ_0.

Let us first prove the following formula. If W is a piece of stable manifold, v is a smooth vector field on a neighborhood of W and $\varphi \in \mathcal{C}_0^1(W)$, then

$$\int_W \varphi L_v \alpha_0 = -\int_W L_{v^s}\varphi \cdot \alpha_0. \tag{8.2}$$

Notice that $L_{v^s}\varphi$ makes sense since v^s is not differentiated here. To prove this, for large n let $v^{s,n}$ and $v^{u,n}$ be approximations of v^s and v^u as constructed in footnote

9. Then

$$\int_W \varphi L_{v^{u,n}}\alpha_0 = \varrho^{-n}\int_W \varphi L_{v^{u,n}}(\mathcal{L}_0^n\alpha_0) = \varrho^{-n}\int_{T^{-n}W^{(n)}} \varphi\circ T_0^n L_{T_0^{*n}v^{u,n}}(\pi_n e^{S_n\phi}\alpha_0).$$

Since $T_0^{*n}v^{u,n}$ has norm at most $C\lambda^{-n}$, this last integral is bounded by

$$C\varrho^{-n}\varrho_n^n\lambda^{-n} \le C\lambda^{-n}, \tag{8.3}$$

which tends to 0 when $n\to\infty$. Hence,

$$\int_W \varphi L_v\alpha_0 = \int_W \varphi L_{v^{u,n}}\alpha_0 - \int_W L_{v^{s,n}}\varphi\cdot\alpha_0 \to -\int_W L_{v^s}\varphi\cdot\alpha_0. \tag{8.4}$$

This proves (8.2). Together with the formula (7.2) for the fixed point of the dual operator, we get for any smooth function φ

$$\ell_0(\varphi L_v\alpha_0) = -\mu_0(L_{v^s}\varphi) - \mu_0\left(\varphi\sum_{n=0}^{\infty} L_{v^s}(\phi_0\circ T_0^n)\right). \tag{8.5}$$

Let α_λ be the eigenfunction of the operator $\mathcal{L}_\lambda$ associated to T_λ and the potential ϕ_λ, normalized so that $\ell_0(\alpha_\lambda)=1$. Let ℓ_λ be the corresponding eigenfunction of the dual operator, with $\ell_\lambda(\alpha_\lambda)=1$. The measure μ_λ is given by $\mu_\lambda(\varphi)=\ell_\lambda(\varphi\alpha_\lambda)$. The derivative at 0 of $\mathcal{L}_\lambda\alpha$ is

$$\mathcal{L}_0'\alpha = L_{v_0}(\mathcal{L}_0\alpha) + \mathcal{L}_0(\phi_0'\alpha). \tag{8.6}$$

Differentiating the equation $\mathcal{L}_\lambda\alpha_\lambda = e^{h_\lambda}\alpha_\lambda$, we get

$$\alpha_0' = e^{-h_0}\mathcal{L}_0\alpha_0' + L_{v_0}\alpha_0 + \phi_0'\circ T_0^{-1}\alpha_0 - h_0'\alpha_0. \tag{8.7}$$

Applying ℓ_0 to this equation, we get $h_0' = \mu_0(\phi_0') + \ell_0(L_{v_0}\alpha_0)$. By (8.5) applied to $\varphi=1$, we obtain $h_0'=\mu_0(A)$.

Since $\ell_0(\alpha_\lambda)=1$, we have $\ell_0(\alpha_0')=0$. Therefore, $(e^{-h_0}\mathcal{L}_0)^n\alpha_0'$ converges to 0 exponentially fast. We can therefore iterate (8.7) and get

$$\alpha_0' = \sum_{k=0}^{\infty}(e^{-h_0}\mathcal{L}_0)^k\big[L_{v_0}\alpha_0 + (\phi_0'\circ T_0^{-1} - h_0')\alpha_0\big]. \tag{8.8}$$

We can use this expression to compute $\ell_0(\varphi\alpha_0')$ when φ is a smooth function. Let $B = -\sum_{n=0}^{\infty} L_{v_0^s}(\phi_0\circ T_0^n)$. Using (8.5) and $h_0'=\mu_0(A)$, we obtain

$$\begin{aligned}\ell_0(\varphi\alpha_0') = \sum_{k=0}^{\infty}\mu_0(\varphi\circ T_0^k(B-\mu_0(B)))\\ - \sum_{k=0}^{\infty}\mu_0(L_{v_0^s}(\varphi\circ T_0^k)) + \sum_{k=1}^{\infty}\mu_0(\varphi\circ T_0^k(\phi_0'-\mu_0(\phi_0'))).\end{aligned} \tag{8.9}$$

For any α, we have $\ell_\lambda(\mathcal{L}_\lambda\alpha) = e^{h_\lambda}\ell_\lambda\alpha$. Differentiating, we get

$$\ell_0'(\alpha) = \ell_0'(e^{-h_0}\mathcal{L}_0\alpha) + \ell_0(L_{v_0}e^{-h_0}\mathcal{L}_0\alpha) + \ell_0((\phi_0'-h_0')\alpha). \tag{8.10}$$

Since $\ell_\lambda(\alpha_\lambda)=1$, we have $\ell_0'(\alpha_0) = -\ell_0(\alpha_0')=0$. Therefore, for any α, $\ell_0'((e^{-h_0}\mathcal{L}_0)^k\alpha)$ converges exponentially fast to 0. Iterating (8.10), we thus get

$$\ell_0'(\alpha) = \sum_{k=0}^{\infty}\ell_0(L_{v_0}(e^{-h_0}\mathcal{L}_0)^{k+1}\alpha) + \ell_0((\phi_0'-h_0')(e^{-h_0}\mathcal{L}_0)^k\alpha). \tag{8.11}$$

Applying this equation to $\alpha = \varphi\alpha_0$ where φ is a smooth function, and using $L_{v_0}\varphi = L_{v_0^s}\varphi + L_{v_0^u}\varphi$ as well as (8.5), we get

$$(8.12) \quad \ell_0'(\varphi\alpha_0) = \sum_{k=-\infty}^{0} \mu_0(\varphi \circ T_0^k(\phi_0' - \mu_0(\phi_0'))) \\ + \sum_{k=-\infty}^{-1} \mu_0(L_{v_0^u}(\varphi \circ T_0^k)) + \sum_{k=-\infty}^{-1} \mu_0(\varphi \circ T_0^k(B - \mu_0(B))).$$

The derivative at 0 of $\mu_\lambda(\varphi) = \ell_\lambda(\varphi\alpha_\lambda)$ is given by $\ell_0'(\varphi\alpha_0) + \ell_0(\varphi\alpha_0')$. Adding (8.12) and (8.9), we obtain the conclusion of the proposition. $\square$

Other quantities that can be shown to depend smoothly from parameters are: the rate of decay of correlations and the associated distributions τ_i (see Theorem 1.2), the variance in the central limit theorem for smooth observables, the rate function in the large deviation for observables (at least in the $\mathcal{C}^\infty$ case), etc.

9. Conformal leafwise measures

This section is formally independent from the rest of the paper, but it is of course written with the hyperbolic setting in mind.

Let X be a locally compact space, endowed with a d-dimensional lamination structure: there exists an atlas $\{(U, \psi_U)\}$ where U is an open subset of X and ψ_U is an homeomorphism from U to a set $D \times K_U$ where D is the unit disk in $\mathbb{R}^d$ and K_U is a locally compact space. Moreover, the changes of charts send leaves to leaves, i.e., $\psi_U \circ \psi_V^{-1}(x, y) = (f(x, y), g(y))$ where defined.

A *continuous leafwise measure* μ is a family of Radon measures on each leaf such that, for all chart (U, ψ_U) as above and all continuous function φ supported in U, $\int_{\psi_U^{-1}(D\times\{y\})} \varphi \,\mathrm{d}\mu$ depends continuously on $y \in K_U$.

Assume that, on each leaf of the lamination, a distance is given, which varies continuously with the leaf (in the sense that, for any chart (U, ψ_U) as above, the map from $D \times D \times K_u$ to $\mathbb{R}$ given by $(x, x', y) \mapsto d(\psi_U^{-1}(x, y), \psi_U^{-1}(x', y))$ is continuous). Consider then an open subset Y of X, with compact closure, and a continuous map $T : Y \to X$ which sends leaves to leaves and expands uniformly the distance: there exist $\kappa > 1$ and $\delta_0 > 0$ such that, whenever x, y are in the same leaf and satisfy $d(x, y) \leq \delta_0$, then $d(Tx, Ty) \geq \kappa d(x, y)$ (in particular, the restriction of T to $B(x, \delta_0)$ is a homeomorphism). Assume that $\Lambda := \bigcap_{n\geq 0} T^{-n}X$ is a compact subset of X.

If $x \in Y$, then T is a homeomorphism on a small ball around x in the leaf containing x. Hence, it is possible to define the pullback $T^*\mu$ of any continuous leafwise measure μ. Our first result is:

Theorem 9.1. *Let μ be a nonnegative continuous leafwise measure, and ν a complex continuous leafwise measure. Assume that there exists a constant $C > 0$ such that, on each leaf, $|\nu| \leq C\mu$. Moreover, assume that there exists a continuous function π, supported in $Y \cap T^{-1}Y$, positive on Λ, Hölder continuous on each leaf, such that $\mu = \pi T^*\mu$ and $\nu = \gamma\pi T^*\nu$ for some $\gamma \in \mathbb{C}$ with $|\gamma| = 1$.*

Then there exist $c \in \mathbb{C}$ and an open subset U of a leaf, containing a point of Λ, such that $\nu = c\mu$ on U.

The proof is essentially a density point argument: there is a small subset where ν is very close to a multiple of μ, and pushing this estimate by T^N for large N we will obtain the result. Technically, the existence of convenient density points will be proved using the martingale convergence theorem. Hence, we will first need to construct good partitions.

Notice first that

$$\text{the leafwise measure } \mu \text{ is supported on } \Lambda. \tag{9.1}$$

Indeed, if a compact set V of a leaf does not intersect Λ, then it can be covered by a finite number of open subsets which are sent in $X\backslash Y$ by some iterate of T. The equation $\mu = \pi T^* \mu$ then shows that μ gives zero mass to each of these open sets.

By compactness of Λ, there exist $\delta \in (0, \delta_0)$ and $\varepsilon_0 > 0$ such that, for any $x \in \Lambda$, the ball $B(x, \delta)$ (in the leaf containing x) is contained in $\{\pi > \varepsilon_0\}$. We fix such a δ until the end of the proof.

We will say that a subset A of a leaf is *good* if it is open with compact closure and $\mu(\partial A) = 0$.

Lemma 9.2. *Let A be a good subset of a leaf, and let $\varepsilon > 0$. There exist good subsets B and $(F_i)_{1\leq i\leq K}$ forming a partition of a full measure subset of A, with $\operatorname{diam}(F_i) \leq \varepsilon$, such that $\mu(B) \leq \mu(A)/2$ and, for all i, there exist $n \in \mathbb{N}$ and $x \in \Lambda$ such that $B(x, \delta/5) \subset T^n F_i \subset B(x, \delta)$.*

Proof. Since $\mu(\partial A) = 0$, there exists $\eta > 0$ such that $V = \{x \in A, d(x, \partial A) \geq \eta\}$ satisfies $\mu(V) \geq \mu(A)/2$. Choose $N > 0$ such that $\kappa^N \varepsilon > \delta$ and $\kappa^N \eta > \delta$.

Define a distance d_N on A by $d_N(x, y) = \sup_{0\leq i\leq N} d(T^i x, T^i y)$. Let $B_N(x, r)$ denote the ball of center x and radius r for the distance d_N. Choose a maximal $\delta/2$-separated set for the distance d_N in $\Lambda \cap V$, say $x_1, \dots, x_k$. Then the balls $B_N(x_i, \delta/4)$ are disjoint, and $T^N(B_N(x_i, \delta/5)) = B(T^N x_i, \delta/5)$. Moreover, $V \cap \Lambda \subset \bigcup B_N(x_i, \delta/2)$.

For each i, there exist $a_i \in (\delta/5, \delta/4)$ with $\mu(\partial B_N(x_i, a_i)) = 0$, and $b_i \in (\delta/2, \delta)$ with $\mu(\partial B_N(x_i, b_i)) = 0$. Define then the sets F_i by induction on i, by

$$F_i = B_N(x_i, b_i) \backslash \left(\bigcup_{j<i} F_j \cup \bigcup_{j>i} \overline{B_N(x_i, a_i)} \right).$$

By construction, the sets F_i are good sets and $B(T^N x_i, \delta/5) \subset T^N F_i \subset B(T^N x_i, \delta)$. Set finally $B = A \backslash \bigcup \overline{F}_i$. The sets F_i cover almost all $V \cap \Lambda$, i.e. almost all V since μ is supported on Λ. This implies that $\mu(B) \leq \mu(A\backslash V) \leq \mu(A)/2$. $\square$

Lemma 9.3. *Let A be a good subset of a leaf, and let $\varepsilon > 0$. There exist good subsets $(F_i)_{i\in\mathbb{N}}$ of A, with $\operatorname{diam}(F_i) \leq \varepsilon$, forming a partition of a full measure subset of A, such that for all $i \in \mathbb{N}$, there exist $n \in \mathbb{N}$ and $x \in \Lambda$ such that $B(x, \delta/5) \subset T^n F_i \subset B(x, \delta)$.*

Proof. It is sufficient to apply inductively Lemma 9.2 to A, then B, and so on. $\square$

Proof of Theorem 9.1. Let us say that a set has "full μ measure" if its intersection with any leaf has full measure in the usual sense. Let $f = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ be the leafwise Radon-Nikodym of ν with respect to μ. It is defined μ almost everywhere. Since $|\nu| \leq C\mu$, it satisfies $|f| \leq C$. The equations $\mu = \pi T^* \mu$ and $\nu = \gamma\pi T^* \nu$ show that, for almost all $x \in \Lambda$, $f(Tx) = \gamma^{-1} f(x)$.

Start from a good set A in a leaf, containing a point of Λ. Applying inductively Lemma 9.3, we obtain a sequence of finer and finer partitions $\mathcal{F}_n$ of a full measure subset of A, such that, for all $F \in \mathcal{F}_n$, there exists $i \in \mathbb{N}$ and $x \in \Lambda$ such that $B(x, \delta/5) \subset T^i F \subset B(x, \delta)$, and with $\operatorname{diam} F \leq 2^{-n}$.

For μ almost every $x \in A$, there is a well defined element $F_n(x) \in \mathcal{F}_n$ containing x. Moreover, the martingale convergence theorem ensures that, for μ almost every x, for all $\varepsilon > 0$,

$$\frac{\mu\{y \in F_n(x) \ : \ |f(y) - f(x)| > \varepsilon\}}{\mu(F_n(x))} \to 0 \text{ when } n \to \infty. \tag{9.2}$$

Fix such a point x. Let $x_n \in \Lambda$ and $i(n) \in \mathbb{N}$ be such that $B(x_n, \delta/5) \subset T^{i(n)} F_n(x) \subset B(x_n, \delta)$. Since π is Hölder continuous and $\pi \geq \varepsilon_0$ on the iterates $T^j F_n(x)$ for all $0 \leq j < i(n)$, there exists a constant C such that, for all $y, z \in F_n(x)$,

$$\prod_{j=0}^{i(n)-1} \pi(T^j y) \leq C \prod_{j=0}^{i(n)-1} \pi(T^j z).$$

Together with (9.2) and the equation $\mu = \pi T^* \mu$, this gives

$$\frac{\mu\{y \in T^{i(n)} F_n(x) \ : \ |f(T^{-i(n)} y) - f(x)| > \varepsilon\}}{\mu(T^{i(n)} F_n(x))} \to 0.$$

Moreover, $f(T^{-i(n)} y) = \gamma^{i(n)} f(y)$, and $\mu(T^{i(n)} F_n(x)) \leq \mu(B(x_n, \delta))$ is uniformly bounded. Hence, for all $\varepsilon > 0$,

$$\mu\{y \in T^{i(n)} F_n(x) \ : \ |f(y) - \gamma^{-i(n)} f(x)| > \varepsilon\} \to 0.$$

Since $T^{i(n)} F_n(x)$ contains the ball $B(x_n, \delta/5)$, we get in particular

$$\mu\{y \in B(x_n, \delta/5) \ : \ |f(y) - \gamma^{-i(n)} f(x)| > \varepsilon\} \to 0. \tag{9.3}$$

Taking a subsequence if necessary, we can assume that x_n converges to a point x' and $\gamma^{-i(n)}$ converges to $\gamma' \in \mathbb{C}$ with $|\gamma'| = 1$. Let φ be a continuous function supported in $B(x', \delta/10)$. Extend it to a continuous function with compact support on nearby leaves. Then (9.3) and the inequality $|f| \leq C$ show that

$$\int_{B(x_n, \delta/5)} \varphi \, \mathrm{d}\nu - f(x)\gamma' \int_{B(x_n, \delta/5)} \varphi \, \mathrm{d}\mu \to 0.$$

By the continuity properties of μ and ν, this implies that

$$\int_{B(x', \delta/10)} \varphi \, \mathrm{d}\nu = f(x)\gamma' \int_{B(x', \delta/10)} \varphi \, \mathrm{d}\mu.$$

Hence, on the ball $B(x', \delta/10)$, we have $\nu = f(x)\gamma' \mu$. □

Proposition 9.4. *Under the assumptions of Theorem 9.1, assume moreover that the map T is topologically mixing on Λ, and that any open set U of a leaf which contains a point of Λ also contains a point of Λ whose orbit is dense. Then there exists $c \in \mathbb{C}$ such that $\nu = c\mu$. In particular, $\gamma = 1$ (or $\nu = 0$).*

Proof. Note first that, if there exists an open subset U of a leaf on which ν vanishes, then $\nu = 0$ and the theorem is trivial. Indeed, there exists $x \in U \cap \Lambda$ whose positive orbit under T is dense in Λ. Let $r \in (0, \delta)$ be such that $B(x, r) \subset U$. The conformality of ν and the expansion properties of T show that, for any $n \in \mathbb{N}$, ν

vanishes on $B(T^n x, r)$. Since ν is continuous, it follows that $\nu = 0$ on Λ. Since ν is supported on Λ, $\nu = 0$.

Assume now that ν is nonzero on each set U as before. Since $|\nu| \leq \mu$, this implies the same property for μ. By Theorem 9.1, there exists an open set U in a leaf, containing a point of Λ, and $c \in \mathbb{C}$ such that $\nu = c\mu$ on U. As above, consider $x \in U \cap \Lambda$ whose orbit is dense, and choose $r \in (0, \delta)$ such that $B(x, r) \subset U$. The conformality of ν and μ shows that, on $B(T^n x, r)$, $\nu = c\gamma^{-n}\mu$. By continuity of the measures, for any $y \in \Lambda$, there exists $f(y) \in \mathbb{C}$ such that $\nu = f(y)\mu$ on $B(y, r)$. Moreover, this $f(y)$ is uniquely defined since μ is nonzero on any ball $B(y, r)$, it depends continuously on $y \in \Lambda$, and it is nonzero by assumption on ν. Finally, $f \circ T = \gamma^{-1} f$.

Since T is topologically mixing, this implies that f is constant and $\gamma = 1$. $\square$

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